3.1. Heat equation. Solve the following IBVP:

$$\begin{aligned} u_t - u_{xx} &= 0 & x \in (0,\pi)\,,\ t>0\,,\\ u(0,t) &= 0 & t>0\,,\\ u(\pi,t) &= 0 & t>0\,,\\ u(x,0) &= \begin{cases} 1 & \text{if } \frac{\pi}{3} \le x \le \frac{2\pi}{3}\,,\\ 0 & \text{if } x<\frac{\pi}{3} \text{ or } \frac{2\pi}{3}<x\,. \end{cases} \end{aligned}$$

Solution: Since we have null Dirichlet boundary conditions, following the discussion given in class (Lecture 3) we use the following Ansatz

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin(nx)\,.$$

Since we only have sin functions as a basis, we compute the Fourier series of the odd extension of $u(x,0)$ with period 2π

$$A_n = \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin(nx)\,dx = \frac{2}{\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin(nx)\,dx = \frac{2}{n\pi}\Big[\cos\big(\frac{n\pi}{3}\big) - \cos\big(\frac{2n\pi}{3}\big)\Big]\,.$$

Notice that the sequence $a_n := \cos\big(\frac{n\pi}{3}\big) - \cos\big(\frac{2n\pi}{3}\big)$ is periodic with period 6 and its values are $a_1 = 1, a_2 = 0, a_3 = -2, a_4 = 0, a_5 = 1, a_6 = 0, \ldots$

Thus, the solution of the original problem is given by

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{a_n}{n} e^{-n^2 t}\sin(nx)\,.$$

3.2. Extreme points of piecewise C^1. Given $T>0$, let $f:(-T,T)\to\mathbb{R}$ be a function such that there exists a partition $\{t_0 = -T < t_1 < \ldots < t_k = T\}$ and a constant $C>0$ such that $f_{|(t_i,t_{i+1})}$ is a C^1 function and $|f(t)| + |f'(t)| \le C$ for all $t_i < t < t_{i+1}$, for $i = 0,1,\ldots,k-1$.

Prove that, for any $i = 0,1,\ldots,k-1$, there exists $f^+(t_i) := \lim_{t\to t_i^+} f(t)$ as well as $f^-(t_i) := \lim_{t\to t_i^-} f(t)$.

Solution: We prove the esistence of $f^+(t_i)$; the proof is analogous for $f^-(t_i)$. For notational convenience, without loss of generality, we assume that $t_i = 0$.

Consider the sequence $f(1), f(\frac{1}{2}), f(\frac{1}{3}), f(\frac{1}{4}), \ldots$. It is a bounded sequence of real numbers (boundedness follows from the assumption $|f| \le C$) and therefore, by compactness, there is a subsequence $f(\frac{1}{n_1}), f(\frac{1}{n_2}), f(\frac{1}{n_3}), \ldots$ which converges to a number x. We will prove that $f^+(0) = x$.

Take any $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $\frac{1}{n_k} < \varepsilon$ and $|f(\frac{1}{n_k}) - x| < \varepsilon$ (we can always find it since $n_k \to \infty$). By the triangle inequality and the fundamental theorem of calculus we have

$$|f(\varepsilon)-x| \leq \left|f(\varepsilon)-f\Big(\frac{1}{n_k}\Big)\right|+\left|f\Big(\frac{1}{n_k}\Big)-x\right| \leq \int_{\frac{1}{n_k}}^{\varepsilon} |f'(t)|\, dt+\varepsilon \leq C\Big|\varepsilon-\frac{1}{n_k}\Big|+\varepsilon \leq (1+C)\varepsilon\,.$$

From the latter inequality, using the definition of limit, one deduce

$$\lim_{\varepsilon \to 0^+} f(\varepsilon) = x\,,$$

which is what we wanted to prove.

3.3. Fourier series for symmetric functions. Let f be a 2π-periodic function. Prove the following statements:

(a) If f is even, i.e. $f(-t) = f(t)\ \forall t$, then the Fourier series of f has the following form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt)\,.$$

(b) If f is odd, i.e. $f(-t) = -f(t)\ \forall t$, then the Fourier series of f has the following form

$$\sum_{k=1}^{\infty} b_k \sin(kt)\,.$$

Solution:

(a) The coefficients b_k have the following form

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin(ks)ds,\ k \geq 1.$$

Since

$$\begin{aligned}
\int_{-\pi}^{\pi} f(s)\sin(ks)ds &= \int_{-\pi}^{0} f(s)\sin(ks)ds + \int_{0}^{\pi} f(s)\sin(ks)ds \\
\text{(substitution } s=-t) &= \int_{0}^{\pi} f(-t)\sin(-kt)dt + \int_{0}^{\pi} f(s)\sin(ks)ds \\
(f \text{ is even }) &= \int_{0}^{\pi} -f(t)\sin(kt)dt + \int_{0}^{\pi} f(s)\sin(ks)ds = 0,
\end{aligned}$$

then $b_k = 0$ for any $k \geq 1$.

(b) The coefficients a_k have the following form

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos(ks)ds,\ k \geq 0.$$

Since

$$\begin{aligned}
\int_{-\pi}^{\pi} f(s)\cos(ks)ds &= \int_{-\pi}^{0} f(s)\cos(ks)ds + \int_{0}^{\pi} f(s)\cos(ks)ds \\
\text{(substitution } s=-t) &= \int_{0}^{\pi} f(-t)\cos(-kt)dt + \int_{0}^{\pi} f(s)\cos(ks)ds \\
(f \text{ is odd }) &= \int_{0}^{\pi} -f(t)\cos(kt)dt + \int_{0}^{\pi} f(s)\cos(ks)ds = 0,
\end{aligned}$$

then $a_k = 0$ for any $k \geq 1$.

3.4. Fourier series I. Compute the real Fourier series (sine/cosine form) of the 2-periodic function

$$f(x) = 1 - x^2, \quad -1 < x < 1\,.$$

Solution: The Fourier series of f is given by the formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\pi x) + b_n\sin(n\pi x)\right]\,.$$

Since f is even, $b_k = 0$ for any $k \geq 1$. We just need to compute a_n, for $n = 0, 1, 2, ...$
For a_0, we have

$$a_0 = \int_{-1}^{1}(1-x^2)\,dx = 2 - \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}.$$

For a_n, we have

$$\begin{aligned}
a_n &= \int_{-1}^{1}(1-x^2)\cos(n\pi x)\,dx \\
&= \int_{-1}^{1}\cos(n\pi x)\,dx - \int_{-1}^{1}x^2\cos(n\pi x)\,dx \\
&= -\left[\frac{\sin(n\pi x)}{n\pi}x^2\right]_{-1}^{1} + \int_{-1}^{1}\sin(n\pi x)\frac{2x}{n\pi}\,dx \\
&= -\left[\frac{\cos(n\pi x)}{n\pi}\frac{2x}{n\pi}\right]_{-1}^{1} + 2\int_{-1}^{1}\frac{\cos(n\pi x)}{(n\pi)^2}\,dx \\
&= -\frac{2}{(n\pi)^2}\left[\cos(n\pi) + \cos(-n\pi)\right] = -\frac{4}{(n\pi)^2}(-1)^n \\
&= \frac{4}{(n\pi)^2}(-1)^{n+1}\,.
\end{aligned}$$

Therefore, the Fourier series of f is

$$\frac{2}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)\right].$$

3.5. Convergent series. Compute the value of the following series

$$\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}.$$

Hint: Compute the Fourier series of 2π-periodic function $f(x)=x^3-\pi^2 x$ for $x\in(-\pi,\pi)$.

Solution: The function $f(x)=x^3-\pi^2x=x(x-\pi)(x+\pi)$ for $x\in(-\pi,\pi)$ is odd, then $a_n=0$ and

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi}(x^3-\pi^2x)\sin(nx)\,dx\\
&= -\frac{2}{\pi}x^3\frac{\cos(nx)}{n}\Big|_0^{\pi}+\frac{2}{\pi}\int_0^{\pi}3x^2\frac{\cos(nx)}{n}\,dx\\
&\quad +\frac{2}{\pi}\pi^2x\frac{\cos(nx)}{n}\Big|_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}\pi^2\frac{\cos(nx)}{n}\,dx\\
&= \frac{6}{\pi n}\int_0^{\pi}x^2\cos(nx)\,dx\\
&= \frac{6}{\pi n}x^2\frac{\sin(nx)}{n}\Big|_0^{\pi}-\frac{6}{\pi n}\int_0^{\pi}2x\frac{\sin(nx)}{n}\,dx\\
&= \frac{12}{\pi n^2}x\frac{\cos(nx)}{n}\Big|_0^{\pi}-\frac{12}{\pi n^2}\int_0^{\pi}\frac{\cos(nx)}{n}\,dx\\
&= \frac{12}{n^3}(-1)^n.
\end{aligned}$$

Therefore, we have

$$f(x)=\sum_{n=1}^{\infty}\frac{12}{n^3}(-1)^n\sin(nx).$$

We insert $x=\pi/2$ in $\sin(n\,\cdot)$ and compute

$$\sin(n\pi/2)=\begin{cases}0 & \text{if}\quad n=2m\\ (-1)^{m+1} & \text{if}\quad n=2m-1.\end{cases}$$

This gives

$$f(\pi/2) = -\frac{3\pi^3}{8} = \sum_{m=1}^{\infty} \frac{12}{(2m-1)^3}(-1)^{2m-1}(-1)^{m+1}$$

and we have

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^3} = \frac{\pi^3}{32}\,.$$