**4.1. Fourier series II.** Let $f:\mathbb{R}\to\mathbb{R}$ be the $2\pi$-periodic even function such that

$$f(x)=e^x \quad \text{for} \quad x\in(0,\pi)\,.$$

Compute the complex Fourier series of $f$. Then, without any additional computation, determine the real Fourier series of the same function by employing the appropriate conversion formulae.

**Solution:** The complex Fourier series of $f$ is an expression of the form

$$f(x)=\sum_{k\in\mathbb{Z}} c_k e^{ikx}\,.$$

The coefficients $c_k$ are given by

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_0^\pi e^x e^{-ikx}\,dx + \frac{1}{2\pi}\int_{-\pi}^0 e^{-x}e^{-ikx}\,dx\\
&= \frac{1}{2\pi}\int_0^\pi e^{(1-ik)x}\,dx + \frac{1}{2\pi}\int_{-\pi}^0 e^{-(1+ik)x}\,dx\\
&= \frac{1}{2\pi}\left(\frac{e^{(1-ik)\pi}-1}{1-ik}+\frac{e^{(1+ik)\pi}-1}{1+ik}\right)\\
&= \frac{e^\pi}{2\pi}\frac{(1+ik)e^{-i\pi k}+(1-ik)e^{i\pi k}}{1+k^2}-\frac{1}{\pi(1+k^2)}\\
&= \frac{e^\pi}{\pi(1+k^2)}\left(\cos(\pi k)+k\sin(\pi k)\right)-\frac{1}{\pi(1+k^2)}\\
&= \frac{e^\pi(-1)^k}{\pi(1+k^2)}-\frac{1}{\pi(1+k^2)}\\
&= \frac{(-1)^k e^\pi-1}{\pi(1+k^2)}\,.
\end{aligned}$$

The real Fourier series of $f$ is an expression of the form

$$f(x)=\sum_{k\geq 0} a_k\cos(kx)+b_k\sin(kx)\,.$$

We have the following relations

$$\begin{aligned}
a_k &= c_k + c_{-k}\,,\\
b_k &= i(c_k-c_{-k})\,.
\end{aligned}$$

Since $f$ is even, $b_k=i(c_k-c_{-k})=0$. For the coefficients $a_k$ we have

$$a_k=c_k+c_{-k}=2c_k=2\frac{(-1)^k e^\pi-1}{\pi(1+k^2)}\,.$$

**4.2. Convergent series.** Let $f_e : \mathbb{R} \to \mathbb{R}$ be the 2-periodic *even* function such that $f_e(x) = x(1-x)$ for $x \in [0,1]$. Let $f_o : \mathbb{R} \to \mathbb{R}$ be the 2-periodic *odd* function such that $f_o(x) = x(1-x)$ for $x \in [0,1]$.

**(a)** Compute the Fourier series of $f_e$.

**(b)** Compute the Fourier series of $f_o$.

**(c)** Use the Fourier series of $f_e$ to compute

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots .$$

**(d)** Use the Fourier series of $f_o$ to compute

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \cdots .$$

**Solution:**

**(a)** The Fourier series of $f_e$ takes the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

Then

$$a_0 = \int_0^1 x(1-x)\ dx = (\frac{x^2}{2} - \frac{x^3}{3})|_{x=0}^{x=1} = \frac{1}{6}$$

and for $n \geq 1$, we have

$$a_n = 2\int_0^1 x(1-x)\cos(n\pi x)\ dx = 2\int_0^1 x\cos(n\pi x)\ dx - 2\int_0^1 x^2 \cos(n\pi x)\ dx.$$

Since

$$\begin{aligned}\int_0^1 x\cos(n\pi x)\ dx &= \underbrace{x\frac{\sin(n\pi x)}{n\pi}|_{x=0}^{x=1}}_{=0} - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\ dx \\ &= \frac{\cos(n\pi x)}{(n\pi)^2}|_{x=0}^{x=1} = \frac{(-1)^n - 1}{n^2\pi^2}\end{aligned}$$

and

$$\begin{aligned}\int_0^1 x^2\cos(n\pi x)\ dx &= \underbrace{x^2\frac{\sin(n\pi x)}{n\pi}\Big|_{x=0}^{x=1}}_{=0} - \int_0^1 2x\frac{\sin(n\pi x)}{n\pi}\ dx \\ &= 2x\frac{\cos(n\pi x)}{(n\pi)^2}\Big|_{x=0}^{x=1} - \int_0^1 2\frac{\cos(n\pi x)}{n^2\pi^2}\ dx \\ &= 2\frac{(-1)^n}{n^2\pi^2} - \underbrace{2\frac{\sin(n\pi x)}{n^3\pi^3}\Big|_{x=0}^{x=1}}_{=0} = 2\frac{(-1)^n}{n^2\pi^2},\end{aligned}$$

we have $a_n = -2\frac{1+(-1)^n}{n^2\pi^2}$. Thus, the Fourier series of $f_e$ is

$$f_e(x) = \frac{1}{6} - 2\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^2\pi^2}\cos(n\pi x). \tag{1}$$

**(b)** The Fourier series of $f_o$ takes the form

$$\sum_{n=1}^{\infty} b_n\sin(n\pi x).$$

We can compute the coefficients $b_n$ for $n\geq 1$ as follows:

$$\begin{aligned}b_n &= 2\int_0^1 f(x)\sin(n\pi x)\ dx = 2\int_0^1 x(1-x)\sin(n\pi x)\ dx \\ &= 2\int_0^1 x\sin(n\pi x)\ dx - 2\int_0^1 x^2\sin(n\pi x)\ dx.\end{aligned}$$

Since

$$\begin{aligned}\int_0^1 x\sin(n\pi x)\,\mathrm{d}x &= -x\frac{\cos(n\pi x)}{n\pi}\Big|_{x=0}^{x=1} + 2\int_0^1\frac{\cos(n\pi x)}{n\pi}\ dx \\ &= -\frac{(-1)^n}{n\pi} + \underbrace{2\frac{\sin(n\pi x)}{n^2\pi^2}}_{=0} = -\frac{(-1)^n}{n\pi}\end{aligned}$$

and

$$\begin{aligned}\int_0^1 x^2\sin(n\pi x)\ dx &= -x^2\frac{\cos(n\pi x)}{n\pi}\Big|_{x=0}^{x=1} + \int_0^1 2x\frac{\cos(n\pi x)}{n\pi}\ dx \\ &= -\frac{(-1)^n}{n\pi} + \underbrace{2x\frac{\sin(n\pi x)}{n^2\pi^2}\Big|_{x=0}^{x=1}}_{=0} - \int_0^1 2\frac{\sin(n\pi x)}{n^2\pi^2}\ dx \\ &= -\frac{(-1)^n}{n\pi} + 2\frac{\cos(n\pi x)}{n^3\pi^3}\Big|_{x=0}^{x=1} \\ &= -\frac{(-1)^n}{n\pi} + 2\frac{(-1)^n-1}{n^3\pi^3},\end{aligned}$$

we have

$$b_n = 4\frac{1-(-1)^n}{n^3\pi^3}.$$

The Fourier series of $f_o$ is

$$\sum_{n=1}^{\infty} 4\frac{1-(-1)^n}{n^3\pi^3}\sin(n\pi x). \tag{2}$$

**(c)** Since $f_e$ is continuous and piecewise differentiable, for every $x \in \mathbb{R}$, the series (1) converges pointwisely to the function $f_e$. Letting $x = 0$ in (1), we get

$$0 = \frac{1}{6} - 2\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^2\pi^2} = \frac{1}{6} - 4\sum_{k=1}^{\infty}\frac{1}{(2k)^2\pi^2},$$

which implies

$$\frac{\pi^2}{6} = \sum_{k=1}^{+\infty}\frac{1}{k^2}.$$

**(d)** Since $f_o$ is continuous and piecewise differentiable, for every $x \in \mathbb{R}$, the series (2) converges pointwisely to the function $f_o$. Letting $x = \frac{1}{2}$ in (2), we obtain

$$\frac{1}{4} = \sum_{n=1}^{+\infty} 4\frac{1-(-1)^n}{n^3\pi^3}\sin\left(n\frac{\pi}{2}\right) = \sum_{n\,\mathrm{odd}}\frac{8}{n^3\pi^3}(-1)^{\frac{n-1}{2}},$$

which implies

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \cdots = \frac{\pi^3}{32}.$$

### 4.3. Heat equations with Neumann boundary conditions.

**(a)** Use a separation of variables Ansatz (i.e., writing $u$ as a sum of solutions of the form $X(x)T(t)$) to solve the following PDE

$$\begin{cases} u_t - 4u_{xx} = 0 & (x,t) \in (0,\pi)\times\mathbb{R}_+ \\ u_x(0,t) = 0 & t \in \mathbb{R}_+ \\ u_x(\pi,t) = 0 & t \in \mathbb{R}_+ \\ u(x,0) = \sin(x) & x \in (0,\pi). \end{cases} \tag{3}$$

**(b)** The PDE (3) describes the heat propagation in an extremely thin channel. Compute the following function

$$U(t) := \frac{1}{\pi}\int_0^\pi u(x,t)\,\mathrm{d}x.$$

One can interpret $U$ as the average temperature of this channel. What can you deduce from $U$?

**(c)** Use a separation of variables Ansatz to solve the following PDE

$$\begin{cases} u_t - 4u_{xx} + u = 0 & (x,t) \in (0,\pi)\times\mathbb{R}_+ \\ u_x(0,t) = 0 & t \in \mathbb{R}_+ \\ u_x(\pi,t) = 0 & t \in \mathbb{R}_+ \\ u(x,0) = \sin(x) & x \in (0,\pi). \end{cases} \tag{4}$$

Compute the average temperature

$$\frac{1}{\pi}\int_0^\pi u(x,t)\,\mathrm{d}x$$

and the deviation of the temperature distribution

$$u(x,t) - \frac{1}{\pi}\int_0^\pi u(x,t)\,\mathrm{d}x.$$

Compare the behavior of the average temperature and of the deviation of the temperature distribution for this problem and the previous one.

**Solution:**

**(a)** With a separation of variables Ansatz $u(x,t) = X(x)T(t)$, we have

$$\begin{cases} X(x)T'(t) &=& 4X''(x)T(t), & \text{for } 0 < x < \pi,\ t > 0, \\ X'(0)T(t) &=& 0, & \text{for } t > 0, \\ X'(\pi)T(t) &=& 0, & \text{for } t > 0. \end{cases}$$

From the first equation and the fact that $T(t)$ is not identically zero, it follows that

$$\frac{4X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

for all $x \in (0,\pi)$ and all $t > 0$. Since this holds only if this value is a constant (which is denoted by $\lambda$), we have the following conditions for $X$

$$\begin{cases} X''(x) &=& \frac{\lambda}{4}X(x), & \text{for } 0 < x < \pi, \\ X'(0) &=& 0, \\ X'(\pi) &=& 0. \end{cases}$$

The solutions to such differential equations are of the form

a) for $\lambda > 0$: $Ae^{\frac{1}{2}\sqrt{\lambda}x} + Be^{-\frac{1}{2}\sqrt{\lambda}x}$,

b) for $\lambda = 0$: $A + Bx$,

c) for $\lambda < 0$: $A\cos(\frac{1}{2}\sqrt{-\lambda}x) + B\sin(\frac{1}{2}\sqrt{-\lambda}x)$.

For the first case, the conditions

$$X'(0) = 0 = X'(\pi)$$

give $A = B = 0$. In the second case, we have $B = 0$ and $A$ can be chosen arbitrarily. In the third case, we have

$$\frac{1}{2}B\sqrt{-\lambda} = 0$$

and

$$-\frac{1}{2}A\sqrt{-\lambda}\sin\left(\frac{1}{2}\sqrt{-\lambda}\pi\right) + \frac{1}{2}B\sqrt{-\lambda}\cos\left(\frac{1}{2}\sqrt{-\lambda}\pi\right) = 0.$$

These relations hold if $\frac{1}{2}\sqrt{-\lambda}\pi$ is a zero of $\sin(\cdot)$, i.e., $\frac{1}{2}\sqrt{-\lambda}\pi = n\pi$ for $n \in \mathbb{Z}$ (note that $n$ and $-n$ give the same function, so it suffices to consider $n > 0$ because $n = 0$ corresponds to the second case which is already considered above). We have $\lambda_n = -4n^2$ for any $n \geq 0$. The associated eigenfunction $X_n$ is (up to a real multiplicative constant) $X_n(x) = \cos(nx)$ and the associated function $T_n$ satisfies the differential equation

$$T_n'(t) = -4n^2 T_n(t),$$

which gives $T_n(t) = ce^{-4n^2 t}$ for all $t \geq 0$.

Now we use the superposition principle. The function $u(x,t) = \sum_{n=0}^{\infty} C_n e^{-4n^2 t}\cos(nx)$ solves our initial value problem with the homogeneous Neumann boundary conditions if and only if

$$\sum_{n=0}^{\infty} C_n \cos(nx) \overset{\text{from (a)}}{=} \sin(x) = \frac{2}{\pi} - \sum_{n=1}^{+\infty} \frac{4}{\pi}\frac{1}{1-4n^2}\cos(2nx), \quad x \in [0,\pi].$$

Therefore, the solution has the form

$$u(x,t) = \frac{2}{\pi} - \sum_{n=1}^{+\infty} \frac{4}{\pi}\frac{1}{1-4n^2} e^{-4n^2 t}\cos(2nx), \quad x \in [0,\pi].$$

**(b)** We compute

$$\begin{aligned}
U(t) &= \frac{1}{\pi}\int_0^{\pi} \frac{2}{\pi} - \sum_{n=1}^{+\infty} \frac{4}{\pi}\frac{1}{1-4n^2}\mathrm{e}^{-4n^2t}\cos(2nx)\mathrm{d}x \\
&= \frac{2}{\pi} - \sum_{n=1}^{+\infty} \frac{4}{\pi^2}\frac{1}{1-4n^2}\mathrm{e}^{-4n^2t}\int_0^{\pi}\cos(2nx)\,\mathrm{d}x \\
&= \frac{2}{\pi} - \sum_{n=1}^{+\infty} \frac{4}{\pi^2}\frac{1}{1-4n^2}\mathrm{e}^{-4n^2t}\underbrace{\left[\frac{\sin(2nx)}{2n}\right]_0^{\pi}}_{=0} = \frac{2}{\pi}.
\end{aligned}$$

We conclude that the mean temperature of the channel is constant with respect to time $t$.

**(c)** With a separation of variables Ansatz $u(x,t) = X(x)T(t)$, we have

$$\begin{cases}
X(x)T'(t) &= 4X''(x)T(t) + X(x)T(t), \quad \text{for } 0 < x < \pi,\ t > 0, \\
X'(0)T(t) &= 0, \quad \text{for } t > 0, \\
X'(\pi)T(t) &= 0, \quad \text{for } t > 0.
\end{cases}$$

From the first equation and the fact that $T(t)$ is not identically zero, it follows that

$$\frac{4X''(x)}{X(x)} = \frac{T'(t)}{T(t)} + 1 := \lambda$$

for all $x \in (0,\pi)$ and all $t > 0$. Since this holds only if this value is a constant (which is denoted by $\lambda$), we have the following conditions for $X$

$$\begin{cases}
X''(x) &= \frac{\lambda}{4}X(x), \quad \text{for } 0 < x < \pi, \\
X'(0) &= 0, \\
X'(\pi) &= 0.
\end{cases}$$

By the same analysis performed above, the eigenvalues are given by $\lambda_n = -4n^2$ for any $n \geq 0$. The associated eigenfunction $X_n$ is (up to a real constant) $X_n(x) = \cos(nx)$ and the associated function $T_n$ satisfies the differential equation

$$T_n'(t) = -(4n^2+1)T_n(t),$$

which gives $T_n(t) = ce^{-(4n^2+1)t}$ for all $t \geq 0$.

Now we use superposition principle. The function $u(x,t) = \sum_{n=0}^{\infty} C_n e^{-(4n^2+1)t}\cos(nx)$ solves our initial value problem with the homogeneous Neumann boundary conditions if and only if

$$\sum_{n=0}^{\infty} C_n\cos(nx) \overset{\text{from (a)}}{=} \sin(x) = \frac{2}{\pi} - \sum_{n=1}^{+\infty}\frac{4}{\pi}\frac{1}{1-4n^2}\cos(2nx), \quad x \in [0,\pi].$$

Therefore, the solution has the form

$$u(x,t) = \frac{2e^{-t}}{\pi} - \sum_{n=1}^{+\infty} \frac{4}{\pi} \frac{1}{1-4n^2} e^{-(4n^2+1)t} \cos(2nx), \quad x \in [0,\pi].$$

It is easy to see that the average temperature is not the same as the solution of (3). Now the average temperature at time $t$ is

$$U(t) = \frac{2e^{-t}}{\pi}.$$

Therefore, we can conclude that because of the presence of the additional term, the average temperature is decreasing and the energy is not conserved.

Now we look at the deviation of the temperature distributions of (3) and (4) and we get

$$\text{deviation}_{(2)} = e^{-t}\text{deviation}_{(1)}.$$

Therefore, we can say the solution of (4) is closer to the average temperature and thus exhibits less deviation from the average temperature.