5.1. Nonhomogeneous to homogeneous. Please study Lesson 6 in Farlow's textbook. Consider the following nonhomogeneous general problem for a function $u : (0,L) \times (0,T) \to \mathbb{R}$

$$\begin{cases} u_t = u_{xx} & \text{in } (0,L) \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in [0,L], \\ \alpha_1 u(0,t) + \beta_1 u_x(0,t) = g_1(t) & \text{for } t \in (0,T), \\ \alpha_2 u(L,t) + \beta_2 u_x(L,t) = g_2(t) & \text{for } t \in (0,T), \end{cases}$$

where $u_0 : [0,L] \to \mathbb{R}$, $g_1, g_2 : (0,T) \to \mathbb{R}$, $\alpha_{1,2}, \beta_{1,2} \in \mathbb{R}$ are given data.

Under the technical assumption $L\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, discuss how to turn it into an equivalent one with homogeneous boundary conditions (the boundary conditions correspond to the last two equations).

Hint: Consider the problem satisfied by $U(x,t) := u(x,t) - A(t)(1 - \frac{x}{L}) - B(t)\frac{x}{L}$, where A, B are two functions (which you shall find) such that $A(t)(1 - \frac{x}{L}) + B(t)\frac{x}{L}$ satisfies the boundary conditions of the problem.

Solution: Let $S(x,t) := A(t)(1 - \frac{x}{L}) + B(t)\frac{x}{L}$, where A, B are functions to be determined. We have

$$S(0,t) = A(t),\ S_x(0,t) = \frac{B(t) - A(t)}{L},\ S(L,t) = B(t),\ S_x(L,t) = \frac{B(t) - A(t)}{L}.$$

Thus, imposing that S satisfies the boundary conditions is equivalent to the linear system (when t is fixed)

$$\begin{cases} (\alpha_1 - \frac{\beta_1}{L})A(t) + \frac{\beta_1}{L}B(t) = g_1(t), \\ -\frac{\beta_2}{L}A(t) + (\alpha_2 + \frac{\beta_2}{L})B(t) = g_2(t), \end{cases}$$

which can be solved provided that the determinant of the associated matrix is nonzero. The determinant is given by

$$(\alpha_1 - \frac{\beta_1}{L})(\alpha_2 + \frac{\beta_2}{L}) + \frac{\beta_1}{L}\frac{\beta_2}{L} = \frac{1}{L}(L\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1) \neq 0.$$

So, A and B are explicit linear combinations of g_1 and g_2; we refrain from writing the explicit formula since it is not particularly useful. The function $U := u - S$ satisfies

$$\begin{cases} U_t = U_{xx} - S_t & \text{in } (0,L) \times (0,T), \\ U(x,0) = U_0(x) - S(x,0) & \text{for } x \in [0,L], \\ \alpha_1 U(0,t) + \beta_1 U_x(0,t) = 0 & \text{for } t \in (0,T), \\ \alpha_2 U(L,t) + \beta_2 U_x(L,t) = 0 & \text{for } t \in (0,T). \end{cases}$$

5.2. Nonhomogeneous heat equation Solve the following problem for $u : [0,1] \times [0,\infty] \to \mathbb{R}$

$$\begin{cases} u_t = u_{xx} & \text{in } (0,1)\times(0,\infty), \\ u(x,0) = x & \text{for } x \in [0,1], \\ u(0,t) = 0 & \text{for } t \in (0,\infty), \\ u(1,t) = \cos(t) & \text{for } t \in (0,\infty). \end{cases}$$

Hint: First transform the problem into one with null boundary conditions, then solve it by expanding it in terms of eigenfunctions.

Solution: Let $U := u - x\cos t$. One can check that U satisfies

$$\begin{cases} U_t = U_{xx} + x\sin(t) & \text{in } (0,1)\times(0,\infty), \\ U(x,0) = 0 & \text{for } x \in [0,1], \\ U(0,t) = 0 & \text{for } t \in (0,\infty), \\ U(1,t) = 0 & \text{for } t \in (0,\infty). \end{cases}$$

To solve this problem, we write the expand the nonhomogeneous with the eigenfunctions of the heat equation with null Dirichlet boundary conditions

$$x\sin(t) = \sin(t)\sum_{n\geq 1}\alpha_n \sin(n\pi x),$$

where $\alpha_n = \frac{2(-1)^n}{\pi n}$. Then, we look for the solution U_n of the simpler problem

$$\begin{cases} (U_n)_t = (U_n)_{xx} + \sin(t)\sin(n\pi x) & \text{in } (0,1)\times(0,\infty), \\ U_n(x,0) = 0 & \text{for } x \in [0,1], \\ U_n(0,t) = 0 & \text{for } t \in (0,\infty), \\ (U_n)(1,t) = 0 & \text{for } t \in (0,\infty), \end{cases}$$

so that the function U is given by $U = \sum_{n\geq 1}\alpha_n U_n$. To find U_n, we employ the Ansatz $U_n(x,t) = V_n(t)\sin(n\pi x)$, which yields the ODE

$$V_n(0) = 0, \quad V_n' + n^2\pi^2 V_n = \sin(t),$$

whose solution is given by

$$V_n(t) = e^{-n^2\pi^2 t}\int_0^t \sin(\tau)e^{n^2\pi^2\tau}d\tau. \tag{1}$$

To compute the integral, notice that for any $a > 0$ one has

$$\begin{aligned}\int_0^t \sin(\tau)e^{a\tau}\,d\tau &= \frac{1}{2i}\int_0^t (e^{i\tau} - e^{-i\tau})e^{a\tau}\,d\tau = \frac{1}{2i}\int_0^t e^{(i+a)\tau} - e^{(-i+a)\tau}\,d\tau \\ &= \frac{1}{2i}\Big(\frac{e^{(i+a)\tau}}{i+a} - \frac{e^{(-i+a)\tau}}{-i+a}\Big) = \frac{e^{at}}{2i}\Big(\frac{-ie^{it} + ae^{it} - ie^{-it} - ae^{-it}}{1+a^2}\Big) \\ &= \frac{e^{at}}{1+a^2}\Big(a\frac{e^{it} - e^{-it}}{2i} - \frac{e^{it} + e^{-it}}{2}\Big) = \frac{e^{at}}{1+a^2}(a\sin(t) - \cos(t)).\end{aligned}$$

Plugging the last identity for $a = n^2\pi^2$ into (1), we obtain

$$V_n(t) = \frac{n^2\pi^2\sin(t) - \cos(t)}{1 + n^4\pi^4}.$$

If we join all the steps, we obtain the solution

$$\begin{aligned}u(x,t) &= x\cos(t) + U(x,t) = x\cos(t) + \sum_{n\geq 1}\alpha_n U_n = x\cos(t) + \sum_{n\geq 1}\alpha_n \sin(n\pi x)V_n(t) \\ &= x\cos(t) + \sum_{n\geq 1}\frac{2(-1)^n}{\pi n}\sin(n\pi x)\frac{n^2\pi^2\sin(t) - \cos(t)}{1+n^4\pi^4}.\end{aligned}$$

5.3. Another variation on the heat equation Solve the following problem for $u : [0,1]\times[0,\infty] \to \mathbb{R}$

$$\begin{cases} u_t = u_{xx} + \sin(\lambda_1 x) & \text{in } (0,1)\times(0,\infty), \\ u(x,0) = 0 & \text{for } x \in [0,1], \\ u(0,t) = 0 & \text{for } t \in (0,\infty), \\ u(1,t) + u_x(1,t) = 0 & \text{for } t \in (0,\infty), \end{cases}$$

where $\lambda_1 > 0$ is the smallest positive solution of $\tan(\lambda) = -\lambda$.

Hint: Use the method of expanding the solution in terms of eigenfunctions.

Solution: The eigenfunctions of the homogeneous problem with respect to the x variable are $\sin(\lambda_n x)$ where λ_n satisfies $\sin(\lambda_n) + \lambda_n\cos(\lambda_n) = 0$. In particular, $\sin(\lambda_1 x)$ is one of the eigenfunctions so the nonhomogeneous term (corresponding to the 'sources') is already expanded in eigenfunctions.

Therefore, proceeding exactly as we did in class in the second part of Lecture 5, we have that the only non-trivial (i.e. not identically zero) contribution corresponds to the Ansatz $u(x,t) = T(t)\sin(\lambda_1 x)$ where $T(t)$ is gotten by solving the Cauchy problem

$$T(0) = 0,\ T' + \lambda_1^2 T = 1$$

whose solution is

$$T(t) = e^{-\lambda_1^2 t} \int_0^t e^{\lambda_1^2 \tau} d\tau = \frac{1 - e^{-\lambda_1^2 t}}{\lambda_1^2}.$$

Therefore the solution of the original problem is given by

$$u(x,t) = \frac{1 - e^{-\lambda_1^2 t}}{\lambda_1^2} \sin(\lambda_1 x).$$