6.1. Integral given the Fourier transform Let f be an integrable function such that

$$\hat{f}(\xi) = \frac{1}{3+4\xi^2}\,.$$

Compute

$$\int_{\mathbb{R}} f(x)\,dx\,.$$

Solution: By definition of Fourier transform, we have

$$\widehat{f}(0) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-i\cdot 0\cdot x}\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{d}x$$

and therefore

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \frac{1}{3}\,.$$

6.2. Moments of function given the Fourier transform Let f be an integrable function such that

$$\hat{f}(\xi) = \frac{3}{5+i\xi}.$$

Compute the following integrals:

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x, \quad \int_{\mathbb{R}} x\,f(x)\,\mathrm{d}x, \quad \int_{\mathbb{R}} x^2\,f(x)\,\mathrm{d}x.$$

Solution: Since

$$\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\hat{f}(\xi) = (-i)^k\widehat{x^k f(x)}(\xi),$$

we have

$$\int_{\mathbb{R}} x^k f(x)\,\mathrm{d}x = \widehat{x^k\,f(x)}(0) = i^k\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\hat{f}(0).$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\,\mathrm{d}x &= \hat{f}(0) = \frac{3}{5},\\
\int_{\mathbb{R}} xf(x)\,\mathrm{d}x &= i\frac{\mathrm{d}}{\mathrm{d}\xi}\hat{f}(0) = \frac{3}{25},\\
\int_{\mathbb{R}} x^2 f(x)\,\mathrm{d}x &= -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\hat{f}(0) = \frac{6}{125}.
\end{aligned}$$

6.3. Tricky integral via Fourier transform With the help of the Fourier transform of $f(x) = \mathrm{e}^{-x^2}$, prove that

$$\int_{\mathbb{R}} x^2 \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Solution: As shown in exercise 6.2, we have

$$\int_{\mathbb{R}} x^2 \mathrm{e}^{-x^2}\,\mathrm{d}x = -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\widehat{e^{-x^2}}(0)$$

and since

$$\widehat{\mathrm{e}^{-x^2}}(\xi) = \sqrt{\pi}\mathrm{e}^{-\frac{\xi^2}{4}}\,,$$

we get

$$\int_{\mathbb{R}} x^2 \mathrm{e}^{-x^2}\,\mathrm{d}x = -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\sqrt{\pi}\mathrm{e}^{-\frac{\xi^2}{4}}\right)\Big|_{\xi=0} = -\sqrt{\pi}\left(-\frac{1}{2}\mathrm{e}^{-\frac{\xi^2}{4}}\right)\Big|_{\xi=0} = \frac{\sqrt{\pi}}{2}\,.$$

6.4. Computing Fourier transform on $\mathbb{R}$. Fix $a > 0$. Compute the Fourier transform of

$$g(x) = \mathrm{e}^{-a|x|} \quad \text{and} \quad h(x) = \operatorname{sign}(x)\mathrm{e}^{-a|x|},$$

where $\operatorname{sign}(x)$ is the sign function, that we here agree to be defined by

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Solution: For the function g, we compute

$$\begin{aligned}
\hat{g}(\xi) &= \int_{\mathbb{R}} \mathrm{e}^{-a|x|-i\xi x}\,\mathrm{d}x = \int_{-\infty}^{0} \mathrm{e}^{ax-i\xi x}\,\mathrm{d}x + \int_{0}^{+\infty} \mathrm{e}^{-ax-i\xi x}\,\mathrm{d}x \\
&= \frac{\mathrm{e}^{ax-i\xi x}}{a-i\xi}\Big|_{-\infty}^{0} + \frac{\mathrm{e}^{-ax-i\xi x}}{-a-i\xi}\Big|_{0}^{+\infty} \\
&= \frac{1}{a-i\xi} - \underbrace{\lim_{R\to-\infty}\frac{\mathrm{e}^{aR-i\xi R}}{a-i\xi}}_{=0} + \underbrace{\lim_{R\to+\infty}\frac{\mathrm{e}^{-aR-i\xi R}}{-a-i\xi}}_{=0} + \frac{1}{a+i\xi} \\
&= \frac{1}{a-i\xi} + \frac{1}{a+i\xi} = \frac{2a}{a^2+\xi^2}.
\end{aligned}$$

For the function h, we compute

$$\begin{aligned}
\hat{h}(\xi) &= \int_{\mathbb{R}} \operatorname{sign}(x) \mathrm{e}^{-a|x|-i\xi x}\,\mathrm{d}x = -\int_{-\infty}^{0} \mathrm{e}^{ax-i\xi x}\,\mathrm{d}x + \int_{0}^{+\infty} \mathrm{e}^{-ax-i\xi x}\,\mathrm{d}x \\
&= -\frac{\mathrm{e}^{ax-i\xi x}}{a-i\xi}\Big|_{-\infty}^{0} + \frac{\mathrm{e}^{-ax-i\xi x}}{-a-i\xi}\Big|_{0}^{+\infty} \\
&= -\frac{1}{a-i\xi} + \underbrace{\lim_{R\to-\infty} \frac{\mathrm{e}^{aR-i\xi R}}{a-i\xi}}_{=0} + \underbrace{\lim_{R\to+\infty} \frac{\mathrm{e}^{-aR-i\xi R}}{-a-i\xi}}_{=0} + \frac{1}{a+i\xi} \\
&= -\frac{1}{a-i\xi} + \frac{1}{a+i\xi} = -\frac{2i\xi}{a^2+\xi^2}.
\end{aligned}$$