7.1. Fourier transform of the indicator function of an interval As in Lecture 7, let $f : \mathbb{R} \to \mathbb{R}$ be the indicator function of $(-1, 1)$, namely

$$f(x) := \begin{cases} 1 & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the convolution product $f * f$.

(b) Without performing any explicit computation (you may use only part **(a)** and the basic properties of the Fourier transform) determine the Fourier transform of the function

$$g(x) := \begin{cases} T + C - x & \text{if } C \leq x \leq C + T, \\ T - C + x & \text{if } C - T \leq x \leq C, \\ 0 & \text{otherwise,} \end{cases}$$

in terms of the positive parameters $T, C > 0$.

Solution:

(a) Since f is the indicator function of $(-1, 1)$, we have

$$\int_E f(y)\, dy = |E \cap (-1, 1)|,$$

where $|\cdot|$ denotes the measure (i.e, the length) of a set. Therefore, we have

$$\begin{aligned} f * f(x) &= \int_{\mathbb{R}} f(y) f(x - y)\, dy = \int_{-1}^{1} f(x - y)\, dy = \int_{x-1}^{x+1} f(y)\, dy \\ &= |(-1, 1) \cap (x - 1, x + 1)|. \end{aligned}$$

If $|x| > 2$, then $|(-1, 1) \cap (x - 1, x + 1)| = 0$. If $|x| \leq 2$, then $|(-1, 1) \cap (x - 1, x + 1)| = 2 - |x|$. Hence, we deduce that

$$f * f(x) = \max(0, 2 - |x|).$$

(b) One can check (for example by drawing a graph or checking the equality separately in the intervals $[-\infty, C - T]$, $[C - T, C]$, $[C, C + T]$, $[C + T, \infty]$) that

$$g(x) = \max(0, 2T - |x - C|) = T \max\Big(0, 2 - |\frac{x - C}{T}|\Big).$$

Therefore, the basic properties of the Fourier transform yield

$$\begin{aligned} \mathcal{F}(g(x))(\xi) &= T e^{-iC\xi} \mathcal{F}\Big(\max\Big(0, 2 - |\frac{x}{T}|\Big)\Big)(\xi) \\ &= T^2 e^{-iC\xi} \mathcal{F}\Big(\max\Big(0, 2 - |x|\Big)\Big)(T\xi). \end{aligned} \tag{1}$$

Under Fourier transform, convolution becomes product, thus using **(a)** we have

$$\mathcal{F}(\max(0, 2 - |x|))(\xi) = \mathcal{F}(f * f)(\xi) = \mathcal{F}(f)(\xi)^2$$

and the Fourier transform of f is given by

$$\mathcal{F}(f)(\xi) = \int_{-1}^{1} e^{-ix\xi} dx = \frac{e^{i\xi} - e^{-i\xi}}{i\xi} = 2\frac{\sin(\xi)}{\xi},$$

thus we obtain

$$\mathcal{F}(\max(0, 2 - |x|))(\xi) = 4\frac{\sin^2 \xi}{\xi^2}. \tag{2}$$

Joining (1) and (2), we get

$$\mathcal{F}(g(x))(\xi) = 4e^{-iC\xi}\frac{\sin^2(T\xi)}{\xi^2}.$$

7.2. Fourier transform. Compute the Fourier transform of the following functions

(a) $f(x) := x^2 e^{-2|x|}$

(b) $g(x) := \sin(2x + 1)e^{-4(x+1)^2}$

Hint: For **(a)**, compute the Fourier transform of $h(x) = e^{-2|x|}$ and find the connection with the Fourier transform of f. For **(b)** use the basic properties of the Fourier transform to reduce the problem to the computation of the Fourier transform of e^{-x^2} (which was performed in class).

Solution:

(a) Let $h(x) := e^{-2|x|}$. Thanks to the basic properties of the Fourier transform, we have

$$\mathcal{F}(x^2 h(x))(\xi) = -\frac{d^2}{d\xi^2}\mathcal{F}(h(x))(\xi).$$

The Fourier transform of h is given by

$$\begin{aligned}\mathcal{F}(h(x))(\xi) &= \int_0^\infty e^{-(2+i\xi)x} dx + \int_{-\infty}^0 e^{(2-i\xi)x} dx \\ &= \frac{1}{2 + i\xi} + \frac{1}{2 - i\xi} \\ &= \frac{4}{4 + \xi^2}.\end{aligned}$$

Therefore, the Fourier transform of $f(x)$ is given by

$$\mathcal{F}(f(x))(\xi) = \mathcal{F}(x^2 h(x))(\xi) = -\frac{d^2}{d\xi^2}\mathcal{F}(h(x)) = -\frac{d^2}{d\xi^2}\frac{4}{4+\xi^2} = \frac{d}{d\xi}\frac{8\xi}{(4+\xi^2)^2}$$
$$= \frac{8}{(4+\xi^2)^2} - \frac{8\xi 4\xi}{(4+\xi^2)^3} = \frac{8(4-3\xi^2)}{(4+\xi^2)^3}.$$

(b) Note the following general identities, valid for any integrable function φ,

$$\mathcal{F}(\varphi(x-a))(\xi) = e^{-ia\xi}\mathcal{F}(\varphi(x))(\xi), \tag{3}$$
$$\mathcal{F}(\varphi(\lambda x))(\xi) = \frac{1}{|\lambda|}\mathcal{F}(\varphi(x))(\xi/\lambda), \tag{4}$$
$$\mathcal{F}(e^{iax}\varphi(x))(\xi) = \mathcal{F}(\varphi(x))(\xi - a). \tag{5}$$

Thanks to (3), we have

$$\mathcal{F}\left(\sin(2x+1)e^{-4(x+1)^2}\right)(\xi) = e^{i\xi}\mathcal{F}\left(\sin(2x-1)e^{-(2x)^2}\right)(\xi),$$

thanks to (4) we have

$$\mathcal{F}\left(\sin(2x-1)e^{-(2x)^2}\right)(\xi) = \frac{1}{2}\mathcal{F}\left(\sin(x-1)e^{-x^2}\right)\left(\frac{\xi}{2}\right),$$

thanks to the linearity of the Fourier transform we have (we are using $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$)

$$\mathcal{F}(\sin(x-1)e^{-x^2})(\xi) = \frac{1}{2i}\Big(e^{-i}\mathcal{F}(e^{ix}e^{-x^2})(\xi) - e^{i}\mathcal{F}(e^{-ix}e^{-x^2})(\xi)\Big),$$

thanks to (5) we have

$$\mathcal{F}(e^{ix}e^{-x^2})(\xi) = \mathcal{F}(e^{-x^2})(\xi-1) \quad \text{and} \quad \mathcal{F}(e^{-ix}e^{-x^2})(\xi) = \mathcal{F}(e^{-x^2})(\xi+1).$$

Using together all the identities we have shown, we get

$$\mathcal{F}(g(x))(\xi) = \frac{e^{i\xi}}{4i}\left[e^{-i}\mathcal{F}(e^{-x^2})\Big(\frac{\xi}{2}-1\Big) - e^{i}\mathcal{F}(e^{-x^2})\Big(\frac{\xi}{2}+1\Big)\right]$$

and, since it was seen in class that the Fourier transform of e^{-x^2} is given by $\sqrt{\pi}e^{-\frac{\xi^2}{4}}$, we deduce

$$\mathcal{F}(g(x))(\xi) = \frac{\sqrt{\pi}e^{i\xi}}{4i}\left[e^{-i}e^{\frac{-(\xi/2-1)^2}{4}} - e^{i}e^{\frac{-\xi/2+1)^2}{4}}\right].$$

7.3. Fourier transform. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous integrable function and let $a \in \mathbb{R}$ be a real number. Define the function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) := g(x+a) - g(x)$. Show that there are infinitely many values $\xi \in \mathbb{R}$ such that $\hat{f}(\xi) = 0$.

Solution: Using the basic properties of the Fourier transform, we obtain

$$\hat{f}(\xi) = e^{i\xi a}\hat{g}(\xi) - \hat{g}(\xi) = (e^{i\xi a} - 1)\hat{g}(\xi)\,.$$

The term $e^{i\xi a} - 1$ has infinitely many zero points, which are

$$\xi_k = \frac{2\pi k}{a} \quad \text{for} \quad k \in \mathbb{Z}$$

and therefore $\hat{f}$ has infinitely many zero points.

7.4. Solving an ODE with the Fourier transform. Find a solution $u : \mathbb{R} \to \mathbb{R}$ to the ODE

$$-u''(x) + u(x) = e^{-|x|}.$$

Hint: Take the Fourier transform of the whole ODE and recall that, for any integrable f, $\mathcal{F}(f * f) = \mathcal{F}(f)^2$.

Solution: Assuming that u is a function that solves the ODE, let us compute the Fourier transform of both sides of the equation (taking for granted that everything is appropriately integrable). Since $\mathcal{F}(u'') = -\xi^2\hat{u}(\xi)$ and $\mathcal{F}(e^{-|x|}) = \frac{2}{1+\xi^2}$ (as shown in **Exercise 6.4**), we have

$$(1+\xi^2)\hat{u} = \mathcal{F}(-u'' + u) = \mathcal{F}(e^{-|x|}) = \frac{2}{1+\xi^2}.$$

Hence, using the formula that links the Fourier transform of a convolution with the product of Fourier transforms, we have

$$\hat{u}(\xi) = \frac{1}{2}\Big(\frac{2}{1+\xi^2}\Big)^2 = \frac{1}{2}\mathcal{F}(e^{-|x|})^2 = \frac{1}{2}\mathcal{F}(e^{-|x|} * e^{-|x|}).$$

Thus (taking the inverse Fourier transform of the last identity), a solution to the ODE is given by

$$u(x) := \frac{1}{2}e^{-|x|} * e^{-|x|} = \int_{\mathbb{R}} e^{-|y|}e^{-|x-y|}dy.$$

Now it remains only to compute the integral explicitly. Let us assume $x \geq 0$ (the case $x < 0$ is analogous) and let us split the integral in three parts:

$$\begin{aligned} \int_{-\infty}^{0} e^{-|y|}e^{-|x-y|}dy &= \int_{-\infty}^{0} e^{y}e^{y-x}dy = e^{-x}\int_{-\infty}^{0} e^{2y}dy = \frac{e^{-x}}{2}, \\ \int_{0}^{x} e^{-|y|}e^{-|x-y|}dy &= \int_{0}^{x} e^{-y}e^{y-x}dy = e^{-x}\int_{0}^{x} 1dy = xe^{-x}, \\ \int_{x}^{\infty} e^{-|y|}e^{-|x-y|}dy &= \int_{x}^{\infty} e^{-y}e^{x-y}dy = e^{x}\int_{x}^{\infty} e^{-2y}dy = \frac{e^{-x}}{2}. \end{aligned}$$

Summing the three contributions, we obtain that for any $x \geq 0$ it holds

$$(e^{-|x|} * e^{-|x|})(x) = (1+x)e^{-x}.$$

Repeating the same computations for $x < 0$ we obtain that for any $x \in \mathbb{R}$ it holds

$$(e^{-|x|} * e^{-|x|})(x) = (1+|x|)e^{-|x|}$$

and thus the solution to the ODE is given by

$$u(x) = \frac{1}{2}(1+|x|)e^{-|x|}.$$

Notice that the function u is smooth for $x \neq 0$ and satisfies $-u''(x) + u(x) = e^{-|x|}$ for any $x \neq 0$. At $x = 0$, one can see that u is differentiable twice (with $u'(0) = 0$ and $u''(0) = -\frac{1}{2}$) and satisfies $-u''(0) + u(0) = 1 = e^{-|0|}$.