**8.1. Separation of variables for the homogeneous wave equation.** Solve the following PDE

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & \text{for } x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\ u_t(x,0) = 0 & \text{for } x \in (0,\pi), \\ u(x,0) = f(x) & \text{for } x \in (0,\pi), \end{cases}$$

where the function $f : [0,\pi] \to \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} x & \text{for} \quad 0 \leq x < \dfrac{\pi}{2}\,, \\ \pi - x & \text{for} \quad \dfrac{\pi}{2} \leq x \leq \pi\,. \end{cases}$$

**Hint:** Proceeding exactly as we did for the heat equation (cf. Lecture 3), you will see that imposing the initial conditions requires you to expand the function $f(x)$ given above only in terms of $\sin(\cdot)$ functions (i.e. employing the basis $\{\sin(nx),\ n = 1,2,\ldots\}$), so that one needs compute the Fourier series *of the odd extension of $f$ with period $2\pi$*.

**Solution:** With the Ansatz $u(x,t) = X(x)T(t)$ we have

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{4T(t)} = -\lambda = \text{constant}.$$

Then we have the ODEs

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T''(t) + 4\lambda T(t) = 0.$$

The boundary condition $u(0,t) = u(\pi,t) = 0$ gives the boundary condition $X(0) = X(\pi) = 0$. Then we have the following solutions

$$\lambda_n = n^2, \quad X_n(x) = \sin(nx) \quad \text{for} \quad n \geq 1\,.$$

Now for $T(t)$, we have

$$T''(t) + 4n^2 T(t) = 0\,.$$

We can compute the following fundamental solutions

$$T_n(t) = \sin(2nt) \quad \text{and} \quad T_n(t) = \cos(2nt)\,.$$

Using the superposition principle, we know that a function $u$ of the following form satisfies the PDE

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(2nt) + B_n \sin(2nt)\right] \sin(nx)\,.$$

The initial condition $u_t(x,0) = 0$ gives

$$u_t(x,0) = \sum_{n=1}^{\infty} 2n\left[-A_n \sin(2n \cdot 0) + B_n \cos(2n \cdot 0)\right]\sin(nx) = 0\,.$$

We deduce that $B_n = 0$ for all $n \geq 1$, thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(2nt)\sin(nx)\,.$$

For the coefficients $A_n$, since we only have sin functions as a basis, we compute the Fourier series of the odd extension of $f$ with period $2\pi$

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin(nx)\,dx \\
&= -\frac{2}{n\pi}x\cos(nx)\Big|_0^{\frac{\pi}{2}} + \frac{2}{n\pi}\int_0^{\frac{\pi}{2}}\cos(nx)\,dx \\
&\quad - \frac{2}{n\pi}(\pi - x)\cos(nx)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{2}{n\pi}\int_{\frac{\pi}{2}}^{\pi}\cos(nx)\,dx \\
&= \frac{4}{n^2\pi}\sin\left(\frac{\pi}{2}n\right)\,.
\end{aligned}$$

Finally, we get the solution

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi}\sin\left(\frac{\pi}{2}n\right)\cos(2nt)\sin(nx)\,.$$

Note that $\sin\left(\frac{\pi}{2}n\right) = 0$ for $n \in 2\mathbb{N}$, then

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\cos(2(2n-1)t)\sin((2n-1)x)\,.$$

**8.2. Separation of variables for the inhomogeneous wave equation.** Solve the following PDE

$$\begin{cases}
u_{tt} - u_{xx} = 1 & \text{for } x \in (0,\pi),\ t > 0, \\
u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\
u_t(x,0) = 0 & \text{for } x \in (0,\pi), \\
u(x,0) = \sin(x) & \text{for } x \in (0,\pi).
\end{cases}$$

**Hint:** We propose two different, but equivalent approaches to solve this problem (in fact, it may be excellent practice for you to solve the problem above in both ways and make sure you get a consistent outcome!).

*First approach: exploit the superposition principle.* You may follow these steps to solve the exercise:

1. Find a particular solution $v(x,t) = v(x)$ which does not depend on the time parameter $t$.
2. Let $w := u - v$ and check that it solves a *homogeneous* wave equation.
3. Employ the separation of variables method (as in **Exercise 8.1**) to find $w$.

*Second approach: perform an eigenfunction expansion, exactly as we had done in class for the inhomogeneous heat equation (cf. Exercise 2 in Lecture 5.)*

In the case of the wave equation, you may benefit from reading Farlow's lesson 20, and then follow the very same strategy (i.e. the same steps).

**Solution:** First we compute a special solution $v$ of this problem:

$$\begin{cases} v_{tt} - v_{xx} = 1 & \text{for } x \in (0,\pi),\ t > 0, \\ v(0,t) = v(\pi,t) = 0 & \text{for } t > 0. \end{cases}$$

We look for a solution $v$ which depends only on $x$, that is $v(x,t) = v(x)$. Then we have $-v''(x) = 1$, so the general solution is given by $v(x) = -\frac{x^2}{2} + bx + c$. Imposing the boundary conditions, we get $b = \frac{\pi}{2}$ and $c = 0$, and therefore

$$v(x) = -\frac{x^2}{2} + \frac{\pi}{2}x.$$

Now let $w = u - v$. We have

$$\begin{cases} w_{tt} - w_{xx} = 0 & \text{for } x \in (0,\pi),\ t > 0, \\ w(0,t) = w(\pi,t) = 0 & \text{for } t > 0, \\ w_t(x,0) = 0 & \text{for } x \in (0,\pi), \\ w(x,0) = \sin(x) + \frac{x^2}{2} - \frac{\pi}{2}x & \text{for } x \in (0,\pi). \end{cases}$$

With the separation Ansatz $w(x,t) = X(x)T(t)$, we have

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = k = \text{constant}.$$

Then we need to solve the ODEs

$$X''(x) - kX(x) = 0 \quad \text{and} \quad T''(t) - kT(t) = 0.$$

The boundary conditions $u(0,t) = u(\pi,t) = 0$ give the boundary conditions $X(0) = X(\pi) = 0$. The initial condition $w_t(x,0) = 0$ gives the initial condition $T'(0) = 0$. For every positive integer $n \geq 1$, we find the solution

$$k = -n^2, \quad X_n(x) = b_n \sin(nx) \quad \text{for } b_n \in \mathbb{R}.$$

Similarly, for $T(t)$, we find the solution

$$T_n(t) = \gamma_n \cos(nt), \quad n \geq 1 \quad \text{for } \gamma_n \in \mathbb{R}.$$

Now we use superposition principle

$$w(x,t) = \sum_{n=1}^{\infty} A_n \cos(nt) \sin(nx)\,.$$

The coefficients $A_n$ can be computed from the initial condition

$$w(x,0) = \sin(x) + \frac{x^2}{2} - \frac{\pi}{2}x.$$

The coefficients of the sin Fourier series of $\frac{x^2}{2} - \frac{\pi}{2}x$ are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} \left( \frac{x^2}{2} - \frac{\pi}{2}x \right) \sin(nx)\,\mathrm{d}x;$$

By partial integration we can compute

$$b_n = \frac{2(-1 + (-1)^n)}{\pi n^3} \quad \forall n \geq 1.$$

Therefore, $w$ is given by

$$w(x,t) = \cos(t)\sin(x) + \sum_{n=1}^{\infty} \frac{2(-1+(-1)^n)}{\pi n^3} \cos(nt)\sin(nx),$$

and $u = w + v$ is given by

$$\begin{aligned} u(x,t) &= v(x,t) + w(x,t) \\ &= -\frac{x^2}{2} + \frac{\pi}{2}x + \cos(t)\sin(x) + \sum_{n=1}^{\infty} \frac{2(-1+(-1)^n)}{\pi n^3} \cos(nt)\sin(nx). \end{aligned}$$

**Alternative solution:** We consider the operator $-\partial_{xx}$ on the interval $(0,\pi)$ with null boundary conditions. Its eigenfunctions are given by $\sin(nx)$ for any positive integer $n \geq 1$. Thus, we employ the ansatz

$$u(x) = \sum_{n\geq 1} a_n(t)\sin(nx).$$

We need to compute the eigenfunction expansion of the inhomogeneous term 1. Since the eigenfunctions are $\sin(nx)$, the eigenfunction expansion coincides with the sin-Fourier series of the odd extension, which is given by $\frac{2}{\pi}\sum_{n\geq 1} b_n \sin(nx)$, where

$$b_n := \int_0^\pi 1\cdot\sin(nx) = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus, we obtain that for any $x \in (0,\pi)$ it holds

$$1 = \frac{4}{\pi}\sum_{n\geq 1,\, n \text{ odd}} \frac{\sin(nx)}{n}.$$

So, the PDE is equivalent to

$$\begin{cases} \sum_{n\geq 1}(a_n''(t) + n^2 a_n(t))\sin(nx) = u_{tt} - u_{xx} = 1 = \frac{4}{\pi}\sum_{n\geq 1,\, n \text{ odd}} \frac{\sin(nx)}{n} & \text{for } x\in(0,\pi),\ t>0, \\ \sum_{n\geq 1} a_n'(0)\sin(nx) = u_t(x,0) = 0 & \text{for } x\in(0,\pi), \\ \sum_{n\geq 1} a_n(0)\sin(nx) = u(x,0) = \sin(x) & \text{for } x\in(0,\pi). \end{cases}$$

In particular we deduce the following family of ODEs:

- If $n = 1$, we have $a_1'' + a_1 = \frac{2}{\pi}$, $a_1(0) = 1$, $a_1'(0) = 0$.
- If $n$ is even, we have $a_n'' + n^2 a_n = 0$, $a_n(0) = 0$, $a_n'(0) = 0$.
- If $n$ is odd and different from 1, we have $a_n'' + n^2 a_n = \frac{4}{\pi n}$, $a_n(0) = 0$, $a_n'(0) = 0$.

The respective solutions are:

- If $n = 1$, $a_1(t) = \cos(t) + \frac{4}{\pi}(1-\cos(t))$.
- If $n$ is even, $a_n(t) = 0$.
- If $n$ is odd and different from 1, $a_n(t) = \frac{4}{\pi n^3}(1-\cos(nt))$.

Hence, the solution $u$ of the problem is given by

$$u(x,t) = \cos(t)\sin(x) + \frac{4}{\pi}\sum_{n\geq 1,\, n \text{ is odd}} \frac{1-\cos(nt)}{n^3}\sin(nx).$$

We leave as an (easy but highly instructive!) exercise for the reader that this function coincides with the solution found with the first approach, which amounts to checking that the Fourier expansion of the odd extension of $v(x) = -x^2/2 + \pi x/2$ to $(0, \pi)$ is equal to

$$\frac{4}{\pi} \sum_{n \geq 1,\, n \text{ is odd}} \frac{\sin(nx)}{n^3},$$

which is indeed the case!