9.1. Homogeneous wave equation on the real line. Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a solution of the PDE

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = e^{-x^2} & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = 0 & \text{for } x \in \mathbb{R}. \end{cases}$$

1. Find the PDE satisfied by the Fourier transform of u with respect to the x variable.

2. Solve the ODE satisfied by $t \mapsto \hat{u}(\xi, t)$ when $\xi \in \mathbb{R}$ is fixed.

3. By computing the inverse Fourier transform of $\hat{u}$, find an explicit formula for u.

Solution: The Fourier transform of the solution u satisfies

$$\widehat{u}_{tt}(\xi, t) + \xi^2 \widehat{u}(\xi, t) = 0.$$

The Fourier transform of the initial condition is

$$\widehat{u}(\xi, 0) = \sqrt{\pi} e^{\frac{-\xi^2}{4}} \quad \text{and} \quad \widehat{u}_t(\xi, 0) = 0\,.$$

Then we have

$$\begin{cases} \widehat{u}_{tt}(\xi, t) + \xi^2 \widehat{u}(\xi, t) = 0, \\ \widehat{u}(\xi, 0) = \sqrt{\pi} e^{\frac{-\xi^2}{4}}, \\ \hat{u}_t(\xi, 0) = 0. \end{cases}$$

If we fix ξ, we have a second-order ODE in t, so the solution is given by

$$\widehat{u}(\xi, t) = A(\xi) \cos(\xi t) + B(\xi) \sin(\xi t).$$

For $A(\xi)$ and $B(\xi)$, by the initial condition we have

$$A(\xi) = \hat{u}(\xi, 0) = \sqrt{\pi} e^{\frac{-\xi^2}{4}} \quad \text{and} \quad \xi B(\xi) = \hat{u}_t(\xi, 0) = 0\,.$$

The solution is then

$$\widehat{u}(\xi, t) = \sqrt{\pi}\, e^{-\xi^2/4} \cos(\xi t)\,.$$

Now we compute the inverse Fourier transform and have

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \sqrt{\pi}\, e^{-\xi^2/4}\cos(\xi t) e^{ix\xi}\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \sqrt{\pi} e^{-\xi^2/4}\frac{e^{i\xi t}+e^{-i\xi t}}{2} e^{ix\xi}\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \sqrt{\pi} e^{-\xi^2/4}\frac{1}{2} e^{i(x+t)\xi}\, d\xi + \frac{1}{2\pi}\int_{\mathbb{R}} \sqrt{\pi} e^{-\xi^2/4}\frac{1}{2} e^{i(x-t)\xi}\, d\xi \\ &= \frac{1}{2}\mathcal{F}^{-1}\left(\sqrt{\pi}e^{-\xi^2/4}\right)(x+t) + \frac{1}{2}\mathcal{F}^{-1}\left(\sqrt{\pi}e^{-\xi^2/4}\right)(x-t) \\ &= \frac{1}{2}\left(e^{-(x+t)^2} + e^{-(x-t)^2}\right). \end{aligned}$$

This solution is the same as the solution given by d'Alembert's formula.

9.2. Inhomogeneous wave equation on the real line. Let $u : \mathbb{R}\times[0,\infty)\to\mathbb{R}$ be a solution of the PDE

$$\begin{cases} u_{tt} - 4u_{xx} = \sin(4t) + x & \text{for } x\in\mathbb{R},\ t>0, \\ u(x,0) = 2x^2 & \text{for } x\in\mathbb{R}, \\ u_t(x,0) = 6\cos(x) & \text{for } x\in\mathbb{R}. \end{cases}$$

1. Find a particular solution v (which does not necessarily satisfy the initial conditions) by employing the Ansatz $v(x,t) = v_1(x) + v_2(t)$.
2. Write down the PDE satisfied by $w := u - v$.
3. Use d'Alembert's formula to find an explicit formula for w and deduce an explicit formula for u.

Solution: We first find a particular solution v of the PDE

$$v_{tt} - 4v_{xx} = \sin(4t) + x.$$

Since on the right-hand-side space and time variables are separated, we are led to make the Ansatz $v(x,t) = v_1(x) + v_2(t)$. The PDE is then equivalent to the couple of ODEs

$$\begin{cases} -4v_1''(x) = x & \text{for } x\in\mathbb{R}, \\ v_2''(t) = \sin(4t) & \text{for } t>0, \end{cases}$$

whose solutions are readily found:

$$v_1(x) = -\frac{x^3}{24} \quad\text{and}\quad v_2(t) = -\frac{\sin(4t)}{16}.$$

We have then the particular solution

$$v(x,t) = -\frac{x^3}{24} - \frac{\sin(4t)}{16}.$$

Now, if u solves the original problem, $w = u - v$ solves the homogeneous problem

$$\begin{cases} w_{tt} - 4w_{xx} = 0 & \text{for } x \in \mathbb{R},\ t > 0, \\ w(x,0) = 2x^2 + \frac{x^3}{24} & \text{for } x \in \mathbb{R}, \\ w_t(x,0) = 6\cos(x) + \frac{1}{4} & \text{for } x \in \mathbb{R}, \end{cases}$$

which we solve by directly using d'Alembert's formula (where the speed of propagation is $c = 2$):

$$w(x,t) = \frac{w(x-2t,0) + w(x+2t,0)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} \left(6\cos(u) + \frac{1}{4}\right) \mathrm{d}u$$

$$\begin{aligned} &= \frac{1}{2}\left(2(x-2t)^2 + \frac{(x-2t)^3}{24} + 2(x+2t)^2 + \frac{(x+2t)^3}{24}\right) + \frac{1}{4}\left[6\sin(u) + \frac{u}{4}\right]_{x-2t}^{x+2t} \\ &= \frac{1}{2}\left(4x^2 + 16t^2 + \frac{1}{24}(2x^3 + 24xt^2)\right) \\ &\quad + \frac{1}{4}\left(6(\sin(x+2t) - \sin(x-2t)) + \frac{1}{4}(x + 2t - x + 2t)\right) \\ &= 2x^2 + 8t^2 + \frac{x^3}{24} + \frac{xt^2}{2} + 3\cos(x)\sin(2t) + \frac{t}{4}, \end{aligned}$$

where we used the trigonometric identity $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos(\alpha)\sin(\beta)$ in the last equality. We conclude that u is

$$u(x,t) = v(x,t) + w(x,t) = 2x^2 + 8t^2 + \frac{xt^2}{2} + \frac{t}{4} - \frac{\sin(4t)}{16} + 3\cos(x)\sin(2t).$$

Alternative solution (only accessible after Lecture 10): We use directly d'Alembert's formula for the inhomogeneous wave equation (based on Duhamel's principle):

$$\begin{aligned} u(x,t) = &\frac{2(x-2t)^2 + 2u(x+2t)^2}{2} \\ &+ \frac{1}{4}\int_{x-2t}^{2+2t} 6\cos(u)\,\mathrm{d}u \\ &+ \frac{1}{4}\int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \sin(4\tau) + u\,\mathrm{d}u\mathrm{d}\tau. \end{aligned}$$

We then compute:

$$\frac{2(x-ct)^2 + 2u(x+ct)^2}{2} = 2x^2 + 8t^2,$$

$$\frac{1}{4}\int_{x-2t}^{2+2t} 6\cos(u)\,\mathrm{d}u = 3\cos(x)\sin(2t),$$

and

$$\frac{1}{4}\int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \sin(4\tau) + u\,\mathrm{d}u\mathrm{d}\tau = \frac{1}{4}\left(t + 2t^2x - \frac{1}{4}\sin(4t)\right).$$

We then conclude that the solution is, as before,

$$u(x,t) = v(x,t) + w(x,t) = 2x^2 + 8t^2 + \frac{xt^2}{2} + \frac{t}{4} - \frac{\sin(4t)}{16} + 3\cos(x)\sin(2t).$$