Mathematik III

Lecture 1 - 23/9/2021

goal: learn language / basic techniques for the study of

$$\underbrace{\text{PDE}}$$

Partial Differential Equations

recall: Ordinary Differential Equations

an $\underbrace{\text{equation}}$ whose unknown is a $\underbrace{\text{function}}$ of one indep. variable

$F(t, u(t), u'(t), \dots, u^{(k)}(t)) = 0$ $\qquad u(t)$

k: order of the equation

Example: Newton's law

$$u''(t) = f(t, u(t), u'(t))$$

Cauchy pb.

$$(*) \begin{cases} u''(t) = f(t, u(t), u'(t)) \\ u(t_0) = u_0 \\ u'(t_0) = u_1 \end{cases}$$

Achtung! also an unknown of the pb.

$\rightarrow$ Cauchy-Lipschitz thm.: has a unique (local) sol.

i.e. $\exists$ open interval $I \ni t_0$ and $u \in C^2(I, \mathbb{R}^d)$ solving $(*)$. Physical interpretation: classical mechanics is deterministic.

Informally: if we don't specify initial position and velocity we get an "infinite family of solutions" ($2d$-dimensional vector space of solutions)

A case study: $(\omega > 0)$

$$\underbrace{u'' + \omega^2 u = 0}_{\text{classic harmonic oscillator}} \xRightarrow{\text{indicial root}} \lambda^2 + \omega^2 = 0$$

$$\lambda = \pm i\omega$$

general sol. $u(t) = A\cos(\omega t) + B\sin(\omega t)$
$(\forall A, B \in \mathbb{R})$

a possible Cauchy pb. : we specify $t_0 = 0$

$u(0) = \ell$

$u'(0) = 0$

then we single out one solution as follows:

$u(0) = \ell = A \implies \boxed{A = \ell}$

$u'(0) = 0 = \omega B \implies \boxed{B = 0}$

$u'(t) = -\omega A \sin(\omega t) + \omega B \cos(\omega t)$

$\leadsto \boxed{u(t) = \ell \cos(\omega t)}$ ∎

Partial Differential Equations

an equation whose unknown is a function of $n \geq 1$ indep. variables

$F(x, u(x), \nabla u(x), \nabla^2 u(x), .., \nabla^k u(x)) = 0$ $\quad u(x_1, ..., x_n)$

K: order of the equation

Examples : 3 2^{nd} order equ. w/ constant coefficients.

① Laplace eq. $\Delta u = 0$

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

steady / static configurations.

② Heat eq. $u_t - \Delta u = 0$

(rmk. $u = u(t, x)$)

diffusion / fast evolution pbs.

③ Wave eq. $u_{tt} - \Delta u = 0$

(rmk. $u = u(t, x)$)

In this course we'll focus on a class $\mathcal{C}$ of PDE that includes all 3 above

2nd order linear PDE
w/ constant coefficients

So:

$$\mathcal{L} u = g \qquad (\#)$$

where, if $u = u(x_1, .., x_n)$

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}\, \partial_{ij}^2 u + \sum_{i=1}^{n} b_i \partial_i u + cu$$

- if $g = 0$ then $(\#)$ is called homogeneous
  else $(\#)$ is called inhomogeneous
- linear means (cf. linear algebra)

$$\mathcal{L}(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \mathcal{L} u_1 + \alpha_2 \mathcal{L} u_2$$

Rmk: if I have an equation like $(\#)$ that is linear and homogeneous $(g=0)$ then $\underbrace{\text{"sum of solutions is a solution"}}$

in Physics: SUPERPOSITION PRINCIPLE

Reality check: what is $\mathcal{L}$ in the 3 examples above?

① Laplace $\qquad \mathcal{L} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

② Heat $\qquad \mathcal{L} = \frac{\partial}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

③ Wave $\qquad \mathcal{L} = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

Classical classification

In the case of functions of 2 independent variables $u(x,y)$

(but the variables could be (t,x) or whatever...)

2nd order linear eq. take the form

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F = G \quad (\$)$$

we have the following terminology:

$(\$)$ is parabolic if $B^2 - 4AC = 0$

$\rightsquigarrow$ heat equation

$(\$)$ is hyperbolic if $B^2 - 4AC > 0$

$\rightsquigarrow$ wave equation

$(\$)$ is elliptic if $B^2 - 4AC < 0$

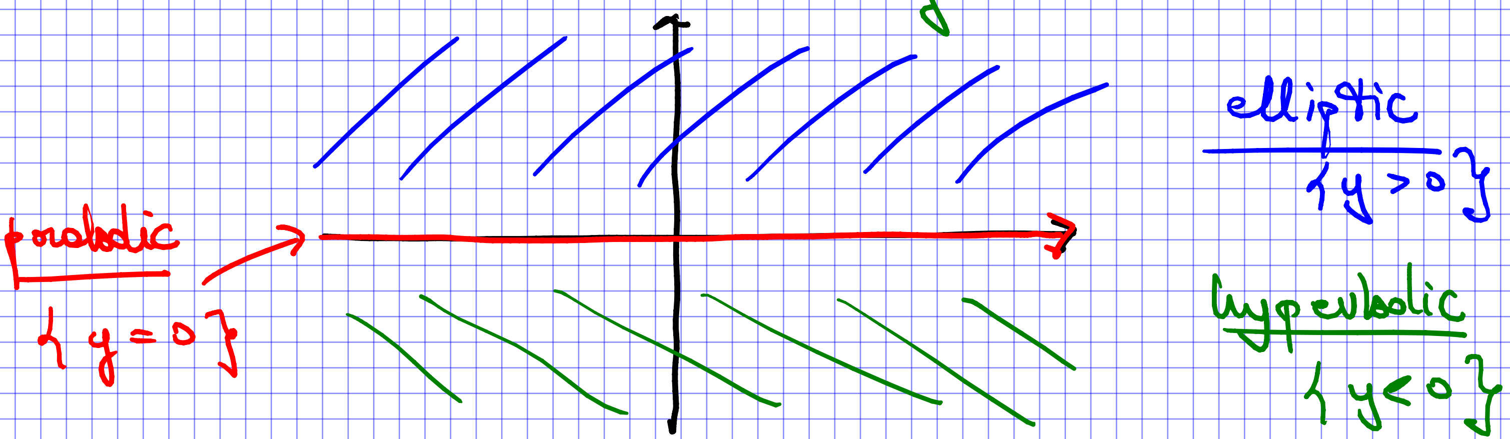
$\rightsquigarrow$ Laplace equation

Note: in general (i.e. if A, B, C are not constant) the type of the equation will depend on $(x,y) \in \mathbb{R}^2$

Example: $y\, u_{xx} + u_{yy} = 0$

$\rightarrow A = y \qquad B = 0 \qquad C = 1$

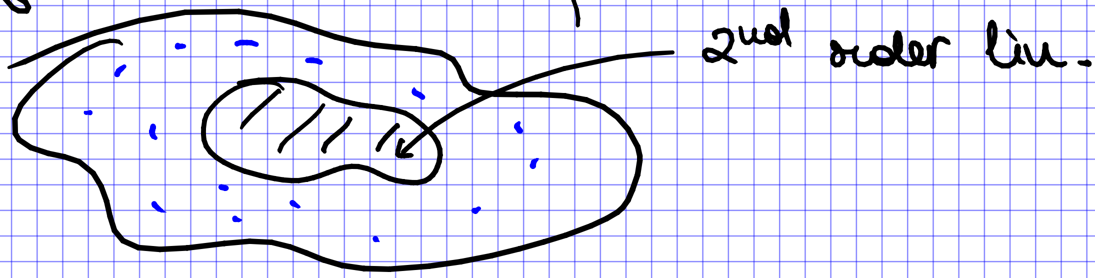
$B^2 - 4AC = -4y$



Rmk. one can give similar definitions for 2^{nd} order linear equations in any number of variables

→ elliptic / parabolic / hyperbolic

Rmk. there are lots of PDE that do not fall in the class of 2^{nd} order linear pbs



for instance:

nonlinear

$$\begin{cases} \Delta u - e^u = 0 & \text{Liouville} \\ \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{Minimal surface eq.} \\ \det(\nabla^2 u) = 0 & \text{Monge–Ampère eq.} \end{cases}$$

$$u_{tt} - u_{xxx} - 6uu_x = 0 \qquad \text{KdV}$$

$$u_t - ku_x = 0 \qquad \text{transport}$$

$- \; - \; - \; -$ Cauchy–Riemann

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Syllabus of the course

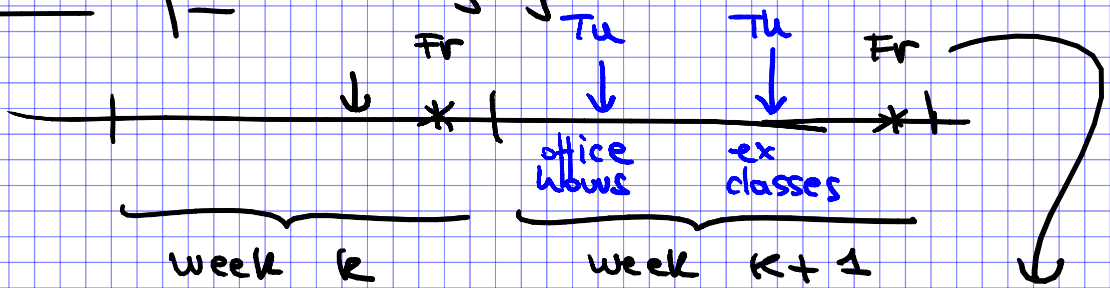
- two general techniques to solve (linear) PDEs
  1. Fourier series + separation of variables
  2. Fourier transform
- basic facts about the 3 model equs above

Diffusion type pb.	Ch. 2
Hyperbolic type pb.	Ch. 3
Elliptic type pb.	Ch. 4

Hierarchy of Resources

primary (abs. necessary)	Lectures	Exercise Classes	Homework
secondary	Textbook	Forum	Office Hours

Additional info: weekly cycle



3 options
→ in class
→ mailbox
→ email

Course coordinator: Federico Glaudo

Exam: 120 min, written