LECTURE 10

25/11/2021

BASIC FACTS ABOUT 1D WAVE PROPAGATION

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < +\infty \quad (\Leftrightarrow \ x \in \mathbb{R}) \\ u(x,0) = f(x) & t \geq 0 \\ u_t(x,0) = g(x) & \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$$

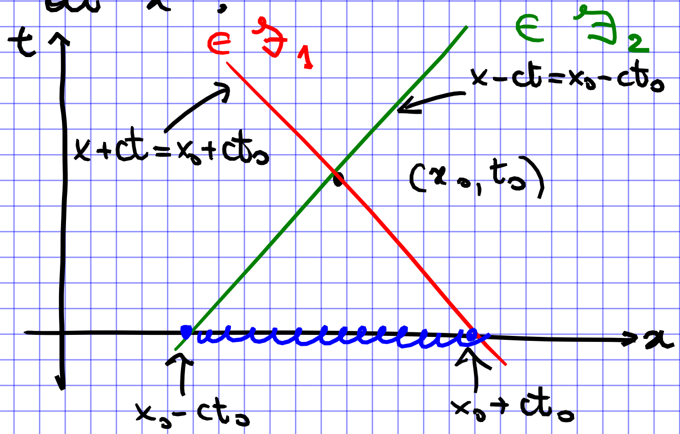
Geometric Meaning

Say: the two families of lines $\mathcal{F}_1 = \{x + ct = \lambda_1, \ \lambda_1 \in \mathbb{R}\}$

$\mathcal{F}_2 = \{x - ct = \lambda_2, \ \lambda_2 \in \mathbb{R}\}$

are called characteristic lines for the equation.

Q. fixed $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$, what is the set of points $x \in \mathbb{R}$ such that $u(x_0, t_0)$ depends on initial data f, g at x?

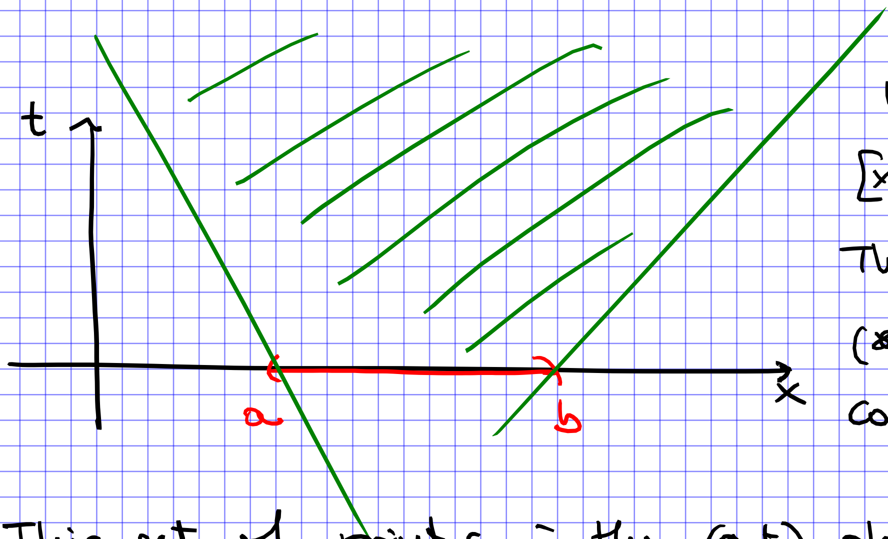


The segment (= closed interval) $[x_0 - ct_0, x_0 + ct_0]$ is called DOMAIN OF DEPENDENCE of u at (x_0, t_0)

Notation: $D(x_0, t_0)$

Q. given an interval $[a, b] \subset \mathbb{R} = \mathbb{R} \times \{0\}$ in (x,t) plane what is the set of points $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ such that $u(x,t)$ is influenced by the value of f, g on $[a,b]$?

Answer: the condition is that the "backward cone" emanating from (x_0, t_0) intersects the segment $[a, b]$



i.e. such points are those for which $[x_0-ct_0, x_0+ct_0] \cap [a,b] = \emptyset$

This happens if and only if (x_0,t_0) is in the FORWARD CONE from $[a,b]$.

This set of points in the (x,t) plane is called REGION (or CONE) of INFLUENCE of $I=[a,b]$

Notation is $\mathcal{C}([a,b])$.

Consequence: if f, g vanish outside $[a,b]$ then u vanishes outside $\mathcal{C}([a,b])$

($\longrightarrow$ wave propagate w/ finite speed, cf. w/ heat equation).

Exercise: Consider the problem
$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = 0 \\ u_t(x,0) = g(x) \end{cases}$$

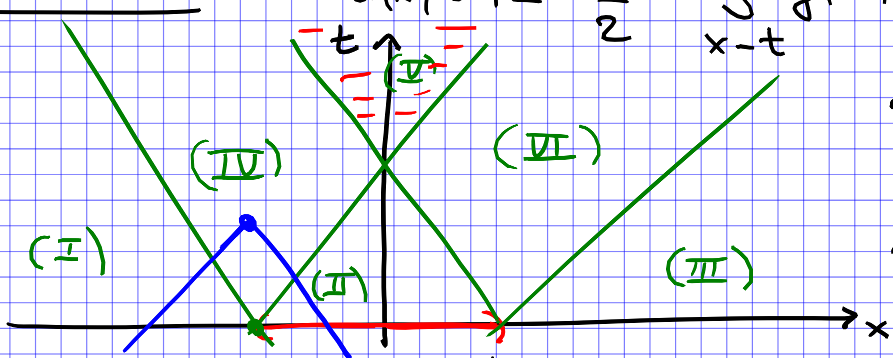
for
$$g(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & \text{else} \end{cases}$$

① determine $u(x,t)$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}_+$

② for any $x \in \mathbb{R}$ determine $\lim_{t\to+\infty} u(x,t)$

③ determine the set of points $(x,t) \in \mathbb{R}\times\mathbb{R}_+$ where the value of $u(x,t)$ is MAX.

Solution:
$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(u)\,du \qquad (c=1)$$



• regions (I) and (III)

$u(x,t) \equiv 0$

• region (IV)

$\{(x,t)\in\mathbb{R}\times\mathbb{R}_+ : \; x+t \in [-1,1], \; x-t \leq -1\}$

$$u(x,t) = \frac{1}{2}\int_{-1}^{x+t} 1\,du = \frac{x+t+1}{2}$$

- region (VI) $\longleftarrow \left\{ (x,t) \in \mathbb{R} \times \mathbb{R}_+ : \begin{array}{l} x-t \in [-1,1] \\ x+t \geq 1 \end{array} \right\}$

$$u(x,t) = \frac{1}{2} \int_{x-t}^{+1} 1 \, du = \frac{1-(x-t)}{2} = \frac{1-x+t}{2}$$

- region (II) $\longleftarrow \left\{ (x,t) \in \mathbb{R} \times \mathbb{R}_+ : \begin{array}{l} x+t \in [-1,1] \\ x-t \in [-1,1] \end{array} \right\}$

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} 1 \, du = \frac{x+t-(x-t)}{2} = t$$

- region (V) $\longleftarrow \left\{ (x,t) \in \mathbb{R} \times \mathbb{R}_+ : \begin{array}{l} x+t \geq 1 \\ x-t \leq -1 \end{array} \right\}$

$$u(x,t) = \frac{1}{2} \int_{-1}^{1} 1 \, du = 1$$

(2) For any fixed x, for t large enough

specifically for $t \geq T(x) = \max\{1+x, 1-x\}$

we have that (x,t) lies in region (V), so

$$t \geq T(x) \implies u(x,t) = 1$$

$$\implies \lim_{t \to +\infty} u(x,t) = 1$$

(3) It follows from (1), going through the 6 cases, that

at any point $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ $\quad 0 \leq u(x,t) \leq 1$

and the MAX value is attained where $u(x,t) = 1$

which is (the closure of) region (V), i.e.

$$\{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : x+t \geq 1, \; x-t \leq -1\}.$$

DUHAMEL'S METHOD (for the wave equation)

↪ Q. how to solve an INHOMOGENEOUS WAVE EQ ?

$$(*)\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

source term (cheap tricks in special cases ↪ Exercise 9.2)

inhomogeneous IC (⇝ D'Alembert)

__Step 0__ : by linearity (superposition principle)

- if u_1 solves $\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$ $(*)_1$

- (and) if u_2 solves $\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$ $(*)_2$

then $u_1 + u_2$ solves the initial problem $(*)$. So we'll only focus on $(*)_1$.

__Step 1__ : (transform sources into retarded initial data)

↪ for each $s \geq 0$ consider the auxiliary system

$$(P_s)\begin{cases} v_{tt} - c^2 v_{xx} = 0 & -\infty < x < +\infty \\ v(x,s) = 0 & (t \geq s) \\ v_t(x,s) = F(x,s) \end{cases}$$

and denote by $u(x,t,s)$ its unique solution.

(why can we solve? We can apply D'Alembert provided we do a time shift. Reduction trick:

set $\boxed{w(x,t) = v(x,t+s)}$ ⇝ $\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = 0 \\ w_t(x,0) = F(x,s) \end{cases}$

D'Al. ⇝ $w(x,t) = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} F(u,s)\,du$

$\rightsquigarrow$ time shift $\quad u(x,t,s) = v(x,t) = w(x, t-s)$

$$= \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} F(u,s)\,du$$

Step 2: (Duhamel's recipe)

A solution to our original problem $(*)_1$ is given by

$$u(x,t) := \int_0^t \underbrace{u(x,t,s)}_{\text{built in Step 1}}\,ds$$

recall $\quad (*)_1 \quad \begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

Check:

- $u(x,0) = \int_0^0 u(x,0,s)\,ds = 0$ ☺

- $u_t(x,0) = \int_0^0 (\text{---}) + \underbrace{u(x,0,0)}_{\text{initial position of the wave solving } (P_s) \text{ for } s=0} = 0$ ☺

$$u_t(x,t) = \left(\int_0^t u_t(x,t,s)\,ds\right) + u(x,t,t)$$

- now we'll check the PDE

$$u(x,t) = \int_0^t u(x,t,s)\,ds \longrightarrow u_{xx}(x,t) = \int_0^t u_{xx}(x,t,s)\,ds$$

differentiate again

$$u_{tt}(x,t) = \left(\int_0^t u_{tt}(x,t,s)\,ds\right) + \underbrace{u_t(x,t,t)}_{\text{Step 1}}$$

$$= \left(\int_0^t u_{tt}(x,t,s)\,ds\right) + F(x,t)$$

Thus

using expressions above

$$(u_{tt}-c^2u_{xx})(x,t)=\left(\int_0^t u_{tt}(x,t,s)\,ds\right)+F(x,t)-c^2\int_0^t u_{xx}(x,t,s)\,ds$$

$$=\int_0^t \underbrace{\left(u_{tt}(x,t,s)-c^2u_{xx}(x,t,s)\right)}_{=0\ \dots\ (\text{by Step 1})}ds + F(x,t)$$

$$= F(x,t)$$ ☺

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Rmk / Advice: solve problem 9.2 using approach above.

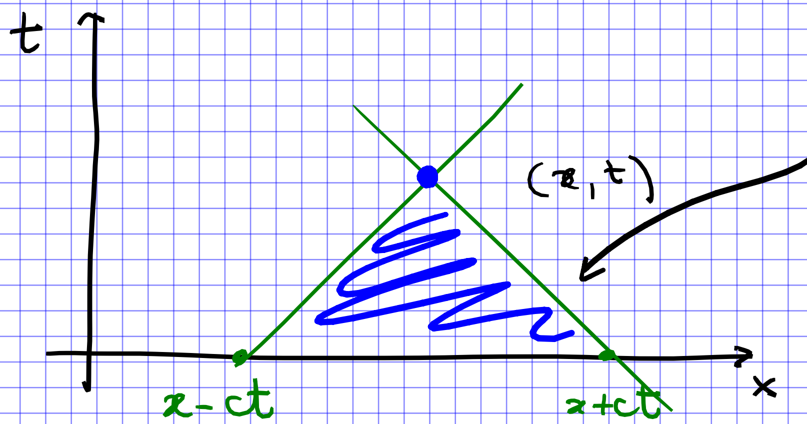
Let's wrap things together: for $(*)_1$

$$u(x,t)=\int_0^t u(x,t,s)\,ds=\int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} F(u,s)\,du\,ds$$

$$=\frac{1}{2c}\iint_T F(u,s)\,du\,ds$$

inhomogeneous D'Alembert

w/ $T := \{(u,s)\in\mathbb{R}^2 : 0\le s\le t,\ |u-x|\le c(t-s)\}$



value $u(x,t)$ depends on F in the whole SOLID triangle