LECTURE 11

2. 12. 2021

Elliptic Boundary Value Problems

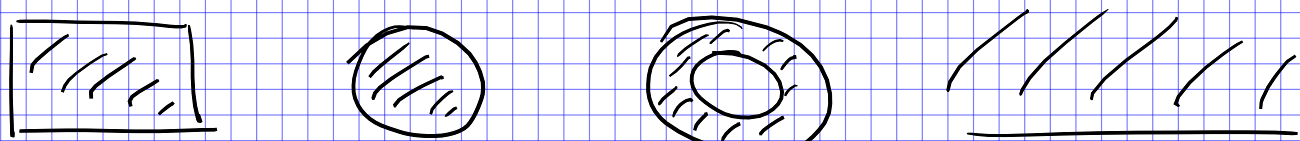
- in $\mathbb{R}^n$, $n \geq 1$ Laplace operator $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$

$\boxed{n=2}$ $\mathbb{R}^2_{(x,y)}$ $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\boxed{n=3}$ $\mathbb{R}^3_{(x,y,z)}$ $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

- Elliptic BVP

$\Omega \subseteq \mathbb{R}^n$ (open) domain, special cases



$(0,a)\times(0,b)$ $B_r(p)$ $B_{r_2}(p) \setminus B_{r_1}(p)$ $\{y>0\} \subset \mathbb{R}^2$

and take
$$\begin{cases} \Delta u = f & \Omega \quad \longleftarrow \text{PDE} \\ \underbrace{\alpha \frac{\partial u}{\partial n} + \beta u = g}_{\text{two important special cases}} & \partial\Omega \quad \longleftarrow \text{BC} \end{cases}$$

$\boxed{u = g}$ Dirichlet BC $\boxed{\frac{\partial u}{\partial n} = g}$ Neumann BC

- general questions: *) EXISTENCE (are there solutions?)
**) UNIQUENESS (only one? finitely many?)

philosophical comment: one can build a "general theory", i.e. theory that applies to "all" domains BUT we'll focus on the specific examples above.

- physical meaning of Dirichlet problem: $(*)\begin{cases} \Delta u = 0 & \Omega \\ u = f & \partial\Omega \end{cases}$

① link w/ heat equation: metal plate $\Omega \subset \mathbb{R}^2$

temperature $u(\underbrace{x,y}_{\text{space}}, t)$ ← time coordinate

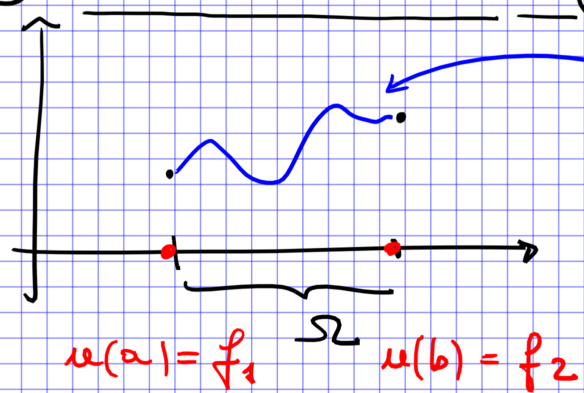
IBVP for temperature

$$\begin{cases} u_t = \alpha^2 \Delta u & x \in \Omega \\ u(x,0) = \varphi(x) & \\ u(x,t) = f(x) & x \in \partial\Omega \end{cases}$$

Steady / asymptotic state: if we wait a large time the temperature profile tends to / settles down to a stationary state $\rightarrow$ $u_t = 0$

$$\rightsquigarrow \begin{cases} \Delta u = 0 \\ u(x,\infty) = f(x) \quad \partial\Omega \end{cases}$$

② **minimise elastic energy:**



$E : C^2(\bar{\Omega}) \longrightarrow \mathbb{R}$

$$E(w) = \frac{1}{2} \int_\Omega |\nabla w|^2$$

Fact:

$\{ u$ minimizes elastic energy E among all functions w/ boundary value given by a known functn $f \}$ $\iff$ $\begin{cases} \Delta u = 0 & \Omega \\ u = f & \partial\Omega \end{cases}$

Another important question (on this topic)

" What are the eigenvalues of Δ as a linear map ? "

Fix $\Omega \subseteq \mathbb{R}^n$

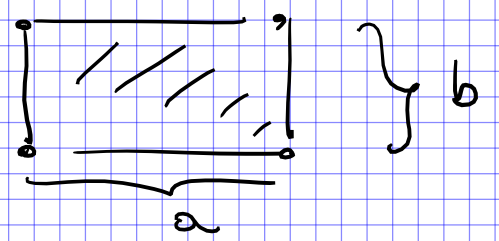
we look for functions u (not identically equal to 0) s.t.

$$\begin{cases} \Delta u = \lambda u & \Omega \\ \alpha \dfrac{\partial u}{\partial n} + \beta u = g & \partial\Omega \end{cases}$$

any $\lambda \in \mathbb{R}$ for which this has a non-trivial (i.e. non identically zero) solution is called EIGENVALUE of Δ (w/ boundary condition $\alpha \frac{\partial u}{\partial n} + \beta u = g$)

Two key exercises:

(1)



- Find eigenvalues of Δ on $R = (0,a)\times(0,b)$ with Dirichlet BC.

Sol. $\begin{cases} \Delta u = \lambda u & R \\ u = 0 & \partial R \end{cases}$

Ansatz: we look for solutions in the special form $u(x,y) = X(x)\,Y(y)$

$\Delta u(x,y) = X''(x)\,Y(y) + X(x)\,Y''(y)$

$\Delta u(x,y) = \lambda u$ takes the concrete form

$$X''(x)\,Y(y) + X(x)\,Y''(y) = \lambda\, X(x)\,Y(y)$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \lambda$$

possible iff $\begin{cases} \dfrac{X''(x)}{X(x)} = \alpha \\ \dfrac{Y''(y)}{Y(y)} = \beta \end{cases}$ w/ $\boxed{\alpha+\beta=\lambda}$

$\begin{cases} X''(x) = \alpha X(x) & x\in(0,a) \\ X(0) = X(a) = 0 \end{cases}$

next: discuss solvability based on sign of α.

- $\alpha = 0 \overset{\text{ODE}}{\Longrightarrow} X(x) = px+q$ (for some $p,q\in\mathbb{R}$)

  $\hookrightarrow \left.\begin{array}{l} X(0) = 0 \\ X(a) = 0 \end{array}\right\} \Rightarrow p=0 \;\; q=0 \quad X(x)\equiv 0$

  no non-trivial sols.

- $\alpha > 0$ $\quad \overset{ODE}{\Longrightarrow} X(x) = p\cosh(\alpha_1 x) + q\sinh(\alpha_1 x)$

$\left(\alpha := \alpha_1^2 \quad \alpha_1 > 0\right)$ $\hookrightarrow$ $\left.\begin{array}{ll} X(0) = 0 \Longrightarrow p = 0 \\ X(a) = 0 \Longrightarrow q = 0 \end{array}\right\} \Longrightarrow X(x) \equiv 0$

no non-trivial sols.

- $\alpha < 0$ $\quad \overset{ODE}{\Longrightarrow} X(x) = p\cos(\alpha_1 x) + q\sin(\alpha_1 x)$

$\left(\alpha = -\alpha_1^2 \quad \alpha_1 > 0\right)$ $\hookrightarrow$ $X(0) = 0 \Longrightarrow p = 0$

$X(a) = 0 \rightsquigarrow \underbrace{q\sin(\alpha_1 a) = 0}$

if we want non-trivial sol.

we must impose $\sin(\alpha_1 a) = 0$

$\boxed{\alpha_1 a = k\pi} \quad k \in \{1, 2, 3, \ldots\}$

Summary: we get non-trivial solutions only for

$$\left\{\begin{array}{l} \alpha_k = \frac{k\pi}{a} \\ k \in \{1, 2, 3, \ldots\} \end{array}\right. \rightsquigarrow X_k(x) = \sin\left(\frac{k\pi x}{a}\right)$$

Totally same story, we get for Y non-trivial sols if

$$\left\{\begin{array}{l} \beta_l = \frac{l\pi}{b} \\ l \in \{1, 2, 3, \ldots\} \end{array}\right. \longrightarrow Y_l(y) = \sin\left(\frac{l\pi y}{b}\right)$$

Conclusion: recalling Ansatz $u(x,y) = X(x)Y(y)$

$$\lambda = \alpha + \beta$$

we get that the eigenvalues (for this pb) are given by

$$\lambda_{k,l} = -\alpha_k^2 - \beta_l^2$$
$$= \boxed{-\left(\frac{k\pi}{a}\right)^2 - \left(\frac{l\pi}{b}\right)^2} \qquad (\#)$$

as one varies $(k, l) \in \{1, 2, 3, \ldots\}^2$

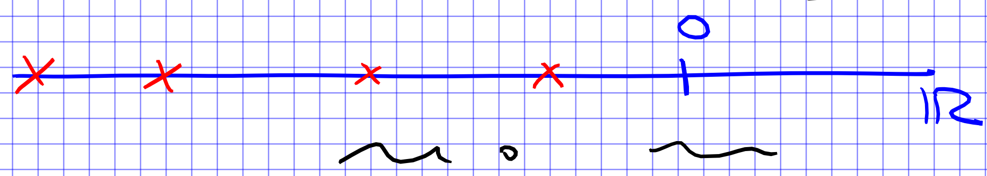
$(k, l) \in \mathbb{N}_*^2$

To pose the bonus question we need a definition.

Def. We'll say that an eigenvalue λ has multiplicity $\mu \geq 1$ if μ is the (real) dimension of the associated eigenspace E_λ. In formula:

$$\mu(\lambda) = \dim_{\mathbb{R}} \left\{ u : \begin{array}{ll} \Delta u = \lambda u & \Omega \\ BC & \partial\Omega \end{array} \right\}$$

Bonus question: [in the special case above] what is the largest eigenvalue that comes w/ multiplicity $\mu \geq 2$ assuming $a = \pi$, $b = 3\pi$.

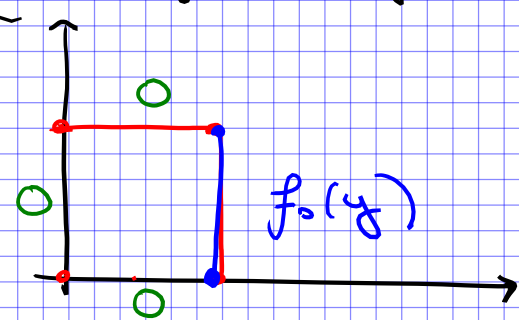


② (cf. pset 11)

$\mathcal{R} = (0,1) \times (0,1) \subset \mathbb{R}^2$

Solve the elliptic BVP
$$\begin{cases} \Delta u(x,y) = 0 & \mathcal{R} \\ u(x,y) = f(x,y) & \partial\mathcal{R} \end{cases}$$

$$f(x,y) = \begin{cases} 0 & y = 0 \\ 0 & y = 1 \\ 0 & x = 0 \\ f_0(y) & x = 1 \end{cases}$$



S. approach via separation of variables. Ultimately look for solutions of the form
$$u(x,y) := \sum_n X_n(x) Y_n(y)$$

One starts from "looking for solutions in product form $X(x)Y(y)$ of the sole PDE", so

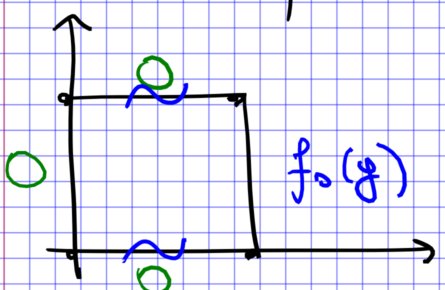
$$X''(x) Y(y) + X(x) Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

usual discussion

$$\begin{cases} \frac{X''}{X} = \sigma \\ \frac{Y''}{Y} = -\sigma \end{cases}$$

$\rightsquigarrow$ to proceed, we need to understand how to employ the BC.



$\longrightarrow$ $y = 0$ $\quad \underbrace{u(x,0)}_{\text{Ansatz}} = f(x,0) = 0$

$$= \sum_n X_n(x) \underbrace{Y_n(0)}_{=0} \quad (\forall n)$$

$\longrightarrow$ $y = 1$ $\quad \underbrace{u(x,1)} = f(x,1) = 0$

$$= \sum_n X_n(x) \underbrace{Y_n(1)}_{=0} \quad (\forall n)$$

So just by using info (i.e. BC) on these two sides I get for Y the ODE problem
$$\begin{cases} Y'' = -\sigma Y \\ Y(0) = 0 \quad Y(1) = 0 \end{cases}$$

Solutions $\boxed{Y_n(y) = \sin(n\pi y)}$ $\quad \sigma_n = (n\pi)^2$

$n \in \{1, 2, 3, \dots\}$

Now, we exploit BC on the two vertical sides:

$\longrightarrow$ $x = 0$ $\quad \underbrace{u(0,y)} = f(0,y) = 0$

$$\sum_n \underbrace{X_n(0)}_{=0} Y_n(y) \quad (\forall n)$$

$\longrightarrow$ $x = 1$ $\quad \underbrace{u(1,y)} = f(1,y) = \overbrace{f_0(y)}^{\text{some given function}}$

$$\sum_n X_n(1) Y_n(y)$$

$$= \sum_n \underline{\underline{X_n(1)}} \sin(n\pi y)$$

$\underbrace{f_0(y)}_{\text{decompose it in } \sin(\cdot)} = \sum_n \underline{\underline{a_n}} \sin(n\pi y)$

$\rightsquigarrow$ $\forall n \in \{1, 2, 3, \dots\}$ $\quad \boxed{X_n(1) = a_n}$

Thus, we wrap things together and get for X the problem:

$$\begin{cases} X'' = \sigma X \\ X(0) = 0 \qquad X(1) = a \end{cases} \qquad \sigma = \sigma_n = (n\pi)^2$$

i.e. $\begin{cases} X_n'' = (n\pi)^2 X_n \\ X_n(0) = 0 \qquad X_n(1) = a_n \end{cases}$ known from the sin Fourier expansion of $f_0(y)$

$\overset{ODE}{\leadsto}$ $X_n(x) = A_n \cosh(n\pi x) + B_n \sinh(n\pi x)$

- impose $X_n(0) = 0 = A_n \quad (\forall n)$
- impose $X_n(1) = a_n = B_n \sinh(n\pi) \quad (\forall n)$

$$\hookrightarrow B_n = \frac{a_n}{\sinh(n\pi)}$$

$$\leadsto X_n(x) = \frac{\sinh(n\pi x) \cdot a_n}{\sinh(n\pi)}$$

So, the solution of our problem is

$$u(x,y) = \sum_n X_n(x) Y_n(y) = \sum_n a_n \frac{\sinh(n\pi x)}{\sinh(n\pi)} \sin(n\pi y)$$