LECTURE 12 9.12.2021

Elliptic BVP on a disc

Review: $\Omega \subseteq \mathbb{R}^n$ domain, two main questions:

① Spectrum ($\Leftrightarrow$ eigenvalues) of Δ under various BC:

$$\begin{cases} \Delta u = \lambda u & \Omega \\ \alpha \dfrac{\partial u}{\partial n} + \beta u = 0 & \partial\Omega \end{cases}$$

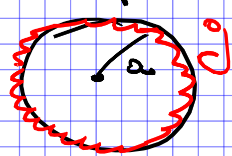
② Solvability of elliptic pbs.

$$\begin{cases} \Delta u = f & \Omega \\ \alpha \dfrac{\partial u}{\partial n} + \beta u = g & \partial\Omega \end{cases}$$

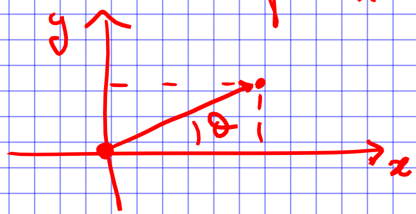
(here: f, g known target is to find u)

Today we'll take $\Omega = D(a) := \{x^2 + y^2 \leq a^2\} \subset \mathbb{R}^2$ and study ② w/ $f = 0$, so look for solutions of

$$\begin{cases} \Delta u = 0 & D(a) \\ u = g & \partial D(a) = S^1(a) \end{cases}$$



Method: separation of variables ⊕ polar coordinates



$$\Delta u \overset{(*)}{=} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

exercise 2.2

where $u = u(r, \theta)$

boundary value is given by $g = g(\theta)$ w/ $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}$

e.g. $-\pi < \theta \leq \pi$

How to solve? $u(r, \theta) = \sum_n R_n(r)\,\Theta_n(\theta)$

Find "elementary blocks" so $R(r)$ and Θ by studying solutions of the sole PDE in the form $R(r)\Theta(\theta)$.

Using $(*)$, we get

$$\Delta u = R''(r)\,\Theta(\vartheta) + r^{-1} R'(r)\,\Theta(\vartheta) + r^{-2} R(r)\,\Theta''(\vartheta) = 0$$

$$\Leftrightarrow \quad \underbrace{r^2 \frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)}} + \underbrace{\frac{\Theta''(\vartheta)}{\Theta(\vartheta)}} = 0$$

$$\begin{cases} \Theta''(\vartheta) = \alpha\,\Theta(\vartheta) \\ r^2 \dfrac{R''(r)}{R(r)} + r\dfrac{R'(r)}{R(r)} + \alpha = 0 \end{cases}$$

$\alpha > 0$ exponentials

$\alpha = 0$ affine

$\alpha < 0$ trigonometric

by periodicity this is the only relevant case

$\alpha = -\sigma^2$

$$\Theta(\vartheta) = A\cos(\sigma\vartheta) + B\sin(\sigma\vartheta)$$

period $= \dfrac{2\pi}{\sigma}$ ← again by periodicity this must to be a divisor of 2π

this happens if and only if σ is an integer

$\sigma = n \in \{1, 2, 3, \dots\}$

$$\Theta_n(\vartheta) = A_n\cos(n\vartheta) + B_n\sin(n\vartheta)$$

For the radial component, we'll get

$$r^2\frac{R_n''(r)}{R_n(r)} + r\frac{R_n'(r)}{R_n(r)} - n^2 = 0$$

$$R_n''(r) + r^{-1}R_n'(r) - r^{-2}n^2 R_n(r) = 0$$

$n \in \{1, 2, 3, \dots\}$ $\qquad R_n \in \langle r^n, r^{-n}\rangle_{\mathbb{R}}$

$n = 0$ $\qquad R_0 \in \langle 1, \log r\rangle_{\mathbb{R}}$

<u>Rmk</u>. by definition, we look for <u>smooth</u> solutions on $D(0)$ in part. we must throw away those solutions above that are unbounded!

$$R_0(r) = c_0 \qquad (c_0 \in \mathbb{R})$$

$$R_n(r) = c_n r^n \qquad (c_n \in \mathbb{R})$$

So, if we put together the pieces, we have that the general sol to our PDE is of the form

$$u(r,\theta) = \sum_n R_n(r)\,\Theta_n(\theta)$$

$$= \sum_{n \geq 0} C_n r^n \cdot \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

$$= \frac{a_0}{2} + \sum_{n \geq 1} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^n$$

renaming constants

We then need to determine a_n, b_n so to match the BC at $r = a$. Here we must decompose g in Fourier series:

$$g(\theta) = \frac{a_0'}{2} + \sum_{n \geq 1} a_n' \cos(n\theta) + b_n' \sin(n\theta)$$

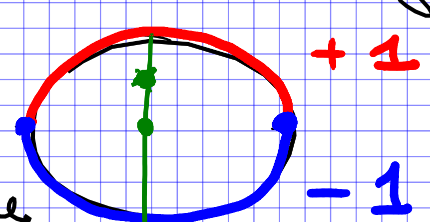
$$\begin{cases} a_n' = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} g(\theta) \cos(n\theta)\, d\theta \\ b_n' = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} g(\theta) \sin(n\theta)\, d\theta \end{cases}$$

Matching $\quad u(a,\theta) = g(\theta)$

gives $\quad a_n' = a_n \cdot a^n \qquad b_n' = b_n \cdot a^n$

$$\boxed{a_n = \frac{a_n'}{a^n}} \qquad \boxed{b_n = \frac{b_n'}{a^n}}$$

Exercise: Let $a = 1$, consider $D(1)$. Conducting plate w/ fixed boundary temperature



Compute the steady state temperature of the plate along the y-axis.

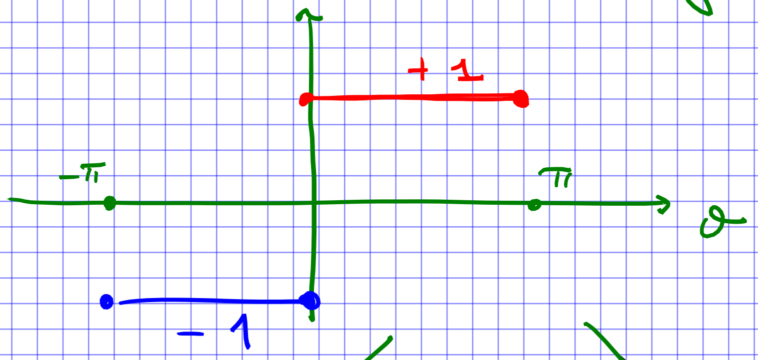
Solution: physical model $\begin{cases} u_t = \alpha^2 \Delta u \\ u(x,0) = \text{—} \\ u(x,t) = g(\theta) = \begin{cases} +1 & 0 \leq \theta < \pi \\ -1 & -\pi < \theta < 0 \end{cases} \end{cases}$

Steady state $\Longrightarrow$ $0 = \cancel{\alpha^2} \Delta u$ $\Longrightarrow \begin{cases} \Delta u = 0 & D \\ u(x,\infty) = g(\theta) & S^1 \end{cases}$

recall (1st hour): general form of the solution

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n\geq 1} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^n$$

task: determine a_n, b_n using BC. $\longleftarrow$ we must expand $g(\theta)$ in Fourier series



$g(\theta)$ is ODD $\Longrightarrow$ $a_n' = 0$ $\forall n = 0,1,2,\ldots$

we must compute the other coeff

$$g(\theta) = \cancel{\frac{a_0'}{2}} + \sum_{n\geq 1} \left(\cancel{a_n' \cos(n\theta)} + \underbrace{b_n' \sin(n\theta)}\right)$$

$$b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta \overset{g \text{ is odd}}{=} \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin(n\theta)\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(n\theta)\,d\theta = \frac{2}{\pi}\left[-\frac{1}{n}\cos(n\theta)\right]\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos(n\pi) + \frac{1}{n}\right] = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n \text{ odd} \end{cases}$$

Thus we get (using the fact that the radius is $= 1$)

$$u(r,\theta) = \sum_{n\geq 1,\, ODD} \frac{4}{n\pi}\sin(n\theta)\, r^n$$

So let's now proceed and answer the question:

y-axis $\rightsquigarrow$ $\theta = \pi/2$ upper half $\longleftarrow$

$\theta = -\pi/2$ lower half

$$u(r, \pi/2) = \sum_{n\geq 1,\, ODD} \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) r^n$$

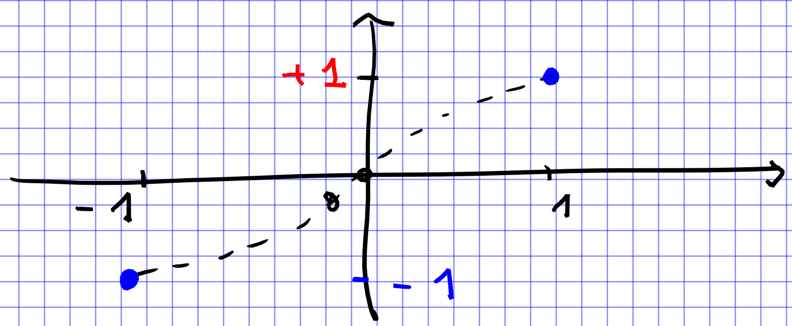
$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} +1 & n = 1,5,9,\ldots \\ -1 & n = 3,7,11,\ldots \end{cases}$$

$$u(r, \pi/2) = \frac{4}{\pi} \underbrace{\left(r - \frac{r^3}{3} + \frac{r^5}{5} - \frac{r^7}{7} + \cdots \right)}_{\arctan(r)}$$

$$\left(\frac{\partial}{\partial r} u(r, \pi/2) = \frac{4}{\pi} \left(1 - r^2 + r^4 - r^6 + \cdots \right) \right.$$

$$\left. = \frac{4}{\pi} \frac{1}{1-(-r^2)} = \frac{4}{\pi} \frac{1}{1+r^2} \cdots \right)$$

$$u(r, \pi/2) = \frac{4}{\pi} \arctan(r)$$



Poisson kernels and mean value property

(1) one can rewrite the general solution

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n \geq 1} (a_n \cos(n\theta) + b_n \sin(n\theta)) r^n$$

in complex form, using

$$\begin{cases} \cos(t) = \dfrac{e^{it} + e^{-it}}{2} \\ \sin(t) = \dfrac{e^{it} - e^{-it}}{2i} \end{cases}$$

plug-in and rearrange

$$u(r, \theta) = \sum_{n \in \mathbb{Z}} c_n r^{|n|} e^{in\theta}$$ w/ coefficients

c_n are determined to match the BC, so

$$u(\text{bdry}) = g(\theta) \implies c_n = a^{-|n|} \int_0^{2\pi} g(\phi) e^{-in\phi} \, d\phi$$

② Q. Can we express our solution to $\begin{cases} \Delta u = 0 & \text{on } D(a) \\ u = g & \text{on } \partial D(a) \end{cases}$ 'directly' in terms of g?

This can be done, and the (unique) sol to ↑ is given by

$$u(r,\theta) \overset{(*)}{=} \frac{1}{2\pi} \int_0^{2\pi} g(\phi)\, P\left(\frac{r}{a}, \theta-\phi\right) d\phi$$

where $$P(q,t) := \frac{1-q^2}{1-2q\cos(t)+q^2} \quad \Big\} \text{ Poisson Kernel}$$

③ Two notable consequences of $(*)$:

(I) <u>mean value principle</u>: '$r=0$' $\longleftrightarrow$ center of the disk

$$P(0,t) = 1 \implies u(\underline{0}) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\, d\phi$$

so "the value of an harmonic function at the center of a disc equals the <u>average</u> of its boundary values"

(II) <u>maximum principle</u>:

$$\min_{\partial\Omega} g \le u(r,\theta) \le \max_{\partial\Omega} g$$