LECTURE 13

16/12/2021

Poisson kernel and applications

$D(a) := \{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 \leq a^2\}$

$$\begin{cases} \Delta u = 0 & D(a) \\ u = g & \partial D(a) \end{cases} \qquad (*)$$

general solution of this problem

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n \geq 1} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^n$$

w/ a_n, b_n computable from g $(\dots)$.

→ can be rewritten in complex form

$$\cos(t) = \frac{e^{it}+e^{-it}}{2} \qquad \sin(t) = \frac{e^{it}-e^{-it}}{2i}$$

if I plug in, for $t = n\theta$, I'll get

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{in\theta} \qquad \text{w/ } c_n \text{ determined}$$

by matching the BC $u = g$ on $\partial D(a)$

$$c_n = a^{-|n|} \frac{1}{2\pi} \int_0^{2\pi} g(\phi)\, e^{-in\phi}\, d\phi$$

__Key result:__ if we define the Poisson kernel to be

$$P(q,t) := \frac{1-q^2}{1-2q\cos(t)+q^2}$$

then the solution of the problem $(*)$ above is

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{g(\phi)}\, \underbrace{P\left(\frac{r}{a}, \theta-\phi\right)}\, d\phi \qquad (P)$$

__Next__: how to get to this formula?

Proof for chemists:

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} c_n\, r^{|n|} e^{in\theta}$$

$$c_n = a^{-|n|} \frac{1}{2\pi} \int_0^{2\pi} g(\phi)\, e^{-in\phi}\, d\phi$$

Plug in:

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} \left(a^{-|n|} \frac{1}{2\pi} \int_0^{2\pi} g(\phi)\, e^{-in\phi}\, d\phi \right) r^{|n|} e^{in\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) \Bigg(\underbrace{\sum_{n\in\mathbb{Z}} \left(\frac{r}{a}\right)^{|n|} e^{in(\theta-\phi)}}_{S} \Bigg) d\phi$$

We now compute S explicitly. We'll use the formula

$$1 + q + q^2 + q^3 + \dots = \frac{1}{1-q} \qquad |q| < 1$$

$$S = \sum_{n\in\mathbb{Z}} \left(\frac{r}{a}\right)^{|n|} e^{in(\theta-\phi)}$$

$$= \sum_{n=0}^{+\infty} \left(\frac{r}{a}\right)^{n} e^{in(\theta-\phi)} + \sum_{n=-\infty}^{-1} \left(\frac{r}{a}\right)^{-n} e^{in(\theta-\phi)}$$

$$= \sum_{n=0}^{+\infty} \left(\frac{r}{a}\right)^{n} e^{in(\theta-\phi)} - 1 + \sum_{n=-\infty}^{0} \left(\frac{r}{a}\right)^{-n} e^{in(\theta-\phi)}$$

$$= \sum_{n=0}^{+\infty} \left(\frac{r}{a}\right)^{n} e^{in(\theta-\phi)} - 1 + \sum_{n=0}^{+\infty} \left(\frac{r}{a}\right)^{n} e^{-in(\theta-\phi)}$$

$$= \frac{1}{1-\left(\frac{r}{a}\right)e^{i(\theta-\phi)}} - 1 + \frac{1}{1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)}}$$

$$= \frac{1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)} + 1 - \left(\frac{r}{a}\right)e^{i(\theta-\phi)} - (\quad)(\quad)}{\left(1-\left(\frac{r}{a}\right)e^{i(\theta-\phi)}\right)\left(1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right)}$$

$$= \frac{1-\left(\frac{r}{a}\right)^2}{1 - 2\left(\frac{r}{a}\right)\cos(\theta-\phi) + \left(\frac{r}{a}\right)^2}$$

So if we replace $\sum$ by $\int$ we'll get

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left(\frac{1-\left(\frac{r}{a}\right)^2}{1-2\left(\frac{r}{a}\right)\cos(\theta-\phi)+\left(\frac{r}{a}\right)^2}\right) d\phi$$

Key consequences:

① **Mean Value Property:** set $r=0$ in (P)

$$u(\text{center}) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} g(\phi)\, d\phi}_{\text{average}}$$

② **Weak maximum principle**

$$\min_{\partial D} g \le u(r,\theta) \le \max_{\partial D} g$$

③ **Uniqueness of solutions**

Consider a problem of the form (#) $\begin{cases} \Delta u = f & D \\ u = g & \partial D \end{cases}$

Then (#) has at most one solution (in fact exactly 1).

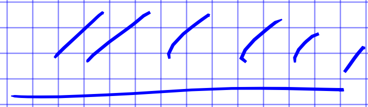
Proof for chemists: let u_1, u_2 both solve (#), set $v := u_2 - u_1$. What problem does v solve?

$$\begin{cases} \Delta v = 0 & D \\ v = 0 & \partial D \end{cases}$$

$\rightsquigarrow$ $v \equiv 0$ $\leftrightsquigarrow$ $u_2 \equiv u_1$.

Laplace equation on a half-plane:

$\mathbb{H} := \{(x,y) \in \mathbb{R}^2 : y > 0\}$

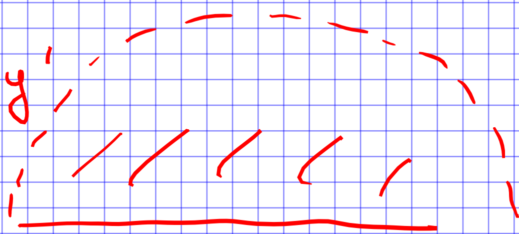


Goal: given $f : \partial\mathbb{H} \to \mathbb{R}$

hp. (f integrable $\int |f(x)| < \infty$, $\hat{f}$ integrable $\int |\hat{f}(x)| < \infty$)

find a solution to
$$\begin{cases} \Delta u = 0 & \mathbb{H} \\ u(x,0) = f(x) & \partial\mathbb{H} \end{cases}$$

such that $\underbrace{\lim_{y \to +\infty} u(x,y) = 0}_{\text{'boundary condition at infinity'}}$ $\quad (\forall x \in \mathbb{R})$



Solution: $\Delta u = u_{xx} + u_{yy}$

We take the Fourier transform of u in the variable x

i.e.
$$\hat{u}(\xi, y) := \int_{-\infty}^{+\infty} u(x,y)\, e^{-ix\xi}\, dx$$

we'll need a differential eq. for $\hat{u}$

$$\hat{u}_{yy} = \int_{-\infty}^{+\infty} u_{yy}(x,y)\, e^{-ix\xi}\, dx \underset{\text{PDE for } u}{=} - \int_{-\infty}^{+\infty} \underbrace{u_{xx}(x,y)}\, e^{-ix\xi}\, dx$$

$$= - \mathcal{F}[u_{xx}(\cdot, y)](\xi)$$

$$= - (i\xi)^2\, \mathcal{F}[u(\cdot, y)](\xi)$$

$$= \xi^2\, \hat{u}(\xi) \longrightarrow \boxed{\hat{u}_{yy} = \xi^2 \hat{u}}$$

$$\longrightarrow \hat{u}(\xi, y) = a(\xi)\, e^{-|\xi| y} + b(\xi)\, e^{|\xi| y}$$

Ansatz: since I'm looking for solutions s.t.

$$u(x,y) \to 0 \qquad y \to +\infty$$

I'll set $b = 0$ and so restrict to

$$\hat{u}(\xi, y) = \underbrace{a(\xi)}\, e^{-|\xi| y}$$

this is determined using BC

$$u(x,0) = f(x) \longrightarrow \hat{u}(\xi, 0) = \hat{f}(\xi)$$

along $\partial \mathbb{H}$ $(x \in \mathbb{R})$ $\qquad = a(\xi)$

thus $\qquad \hat{u}(\xi, y) = \hat{f}(\xi)\, e^{-|\xi| y}$

Now, we apply the inversion formula 'to compute u from $\hat{u}$'

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}(\xi, y)\, e^{ix\xi}\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi)\, e^{-|\xi| y}\, e^{ix\xi}\, d\xi$$

$$= \mathcal{F}^{-1}\left[\hat{f}(\xi) \cdot e^{-|\xi| y} \right]$$

$$= \mathcal{F}^{-1}\left[\hat{f} \right] * \mathcal{F}^{-1}\left[e^{-|\xi| y} \right]$$

$$= f * \left(\frac{1}{\pi} \frac{y}{x^2 + y^2} \right)$$ problem 6.4

$$\left(a * b\, (x) := \int a(x-u)\, b(u)\, du \right)$$

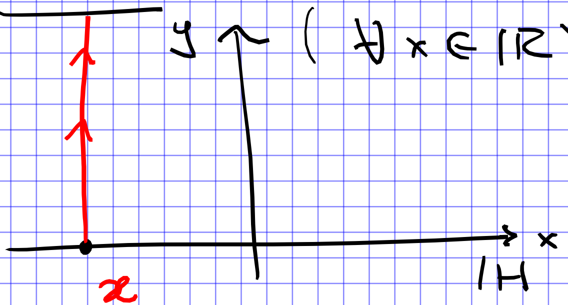
$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(x-s)\, y}{s^2 + y^2}\, ds$$

<u>Summary</u>: we have built a solution to $\begin{cases} \Delta u = 0 & \mathbb{H} \\ u = f & \partial \mathbb{H} \end{cases}$

this is given by $u(x,y) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{+\infty} \frac{f(x-s)\, y}{s^2+y^2}\, ds$

(cultural corner: $k(x,y) = \dfrac{1}{\pi} \dfrac{y}{x^2+y^2}$ is called <u>Cauchy</u> kernel)

<u>Still</u> need to check $\lim\limits_{y \to +\infty} u(x,y) = 0$ $(\forall x \in \mathbb{R})$



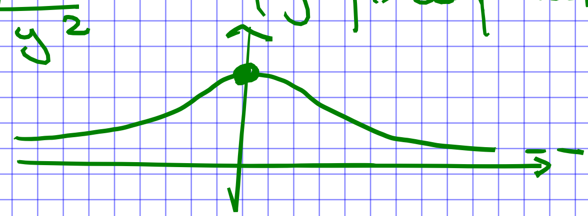
<u>Proof for chemists</u>:

$$|u(x,y)| \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} |f(x-s)| \underbrace{\left| \frac{y}{s^2+y^2} \right|}\, ds$$

$$\leq \frac{1}{\pi} \int_{-\infty}^{+\infty} |f(x-s)| \underbrace{\left\{ \sup_{s \in \mathbb{R}} \frac{y}{s^2+y^2} \right\}}_{\text{computed "differentiating"}} ds$$

$$\mathbb{R} \ni s \longmapsto \underset{w(s)}{} \frac{y}{s^2+y^2} \quad (y \text{ fixed parameter})$$

$$w'(s) = y \left(\frac{-2s}{(s^2+y^2)^2} \right)$$



$$\sup_{s \in \mathbb{R}} \frac{y}{s^2+y^2} = \left(\frac{y}{s^2+y^2} \right) \Bigg|_{s=0} = \frac{1}{y}$$

thus
$$|u(x,y)| \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} |f(x-s)| \frac{1}{y}\, ds$$
$$= \frac{1}{y\pi} \underbrace{\int_{-\infty}^{+\infty} |f(x-s)|\, ds}_{= \int_{-\infty}^{+\infty} |f(s)|\, ds}$$

<u>Conclusion</u>: $|u(x,y)| \leq \dfrac{\int |f(s)|\, ds}{\pi y}$

thus $\lim\limits_{y \to \infty} |u(x,y)| = 0$ ■