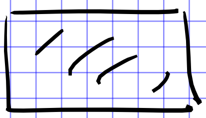
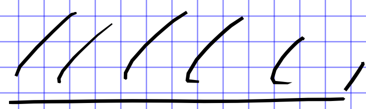
LECTURE 14

23/12/2021

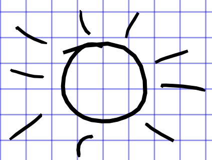
3D Laplace equation in spherical coordinates

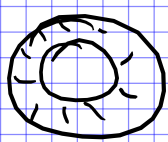
So far we have studied elliptic equations in 2D domains:

now we work w/ spherically symmetric 3D domains:

1. ball  $B^3_R = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le R^2\}$

2. exterior ball  $E^3_R = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \ge R^2\}$

3. 3D annuli  $A^3_{R_1,R_2} = \{(x,y,z) \in \mathbb{R}^3 : R_1^2 \le x^2+y^2+z^2 \le R_2^2\}$

Two types of questions (for any of these domains, say Ω)

i) <u>spectrum of Δ</u> $\longrightarrow$ Helmholtz equation w/ some BC

i.e. find λ s.t. $\begin{cases} \Delta u = -\lambda u & \Omega \\ BC & \partial\Omega \end{cases}$

has <u>non-trivial</u> solutions (u not identically $= 0$)

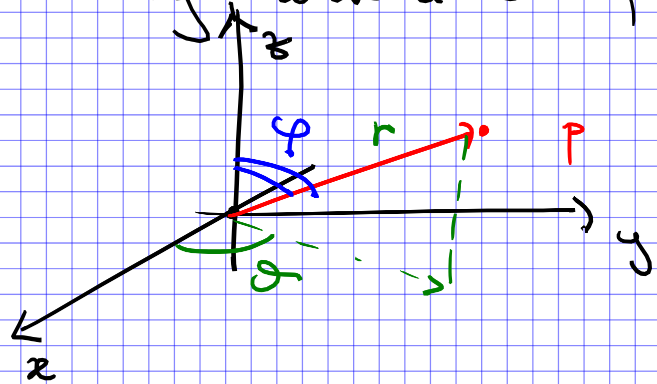
ii) <u>boundary value problems</u>

solve $\begin{cases} \Delta u = f & \Omega \\ BC & \partial\Omega \end{cases}$

Special case: Dirichlet problem for harmonic functions

$$\begin{cases} \Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$$

Today work in 3D spherical coordinates:



$$\begin{cases} x = r \sin\varphi \cos\vartheta \\ y = r \sin\varphi \sin\vartheta \\ z = r\cos\varphi \end{cases}$$

Expression of Laplace operator in 3D spherical coordinates:

$u = u(r, \varphi, \vartheta)$

$$\Delta u \underset{\text{chain rule}}{=} \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial u}{\partial \varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial\vartheta^2}\right]$$

Rmk. If we want to study the questions above in 3D (e.g. for B^3) we should start w/ Ansatz

$$u(r, \varphi, \vartheta) = R(r)\,\Phi(\varphi)\,\Theta(\vartheta)$$

Target of today's lecture: study question ii) above in the special case when the boundary datum g only depends on the latitude, i.e. $g = g(\varphi)$

Concrete special case: solve $\begin{cases} \Delta u = 0 & B^3_1 \\ u = 1 - \cos(2\varphi) & \partial B_1 \end{cases}$

Ansatz: we'll look for sol. $\boxed{u(r, \varphi, \vartheta) = R(r)\,\Phi(\varphi)}$

Solution: compute Δu in spherical coordinates given this Ansatz

$$\Delta u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 R'(r)\,\Phi(\varphi)\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi \cdot R(r)\,\Phi'(\varphi)\right)\right]$$

$$= \frac{1}{r^2}\left[r^2 R''(r)\,\Phi(\varphi) + 2r R'(r)\,\Phi(\varphi) + \frac{R(r)}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\, \Phi'(\varphi)\right)\right]$$

$\Delta u = 0 \iff$

$$\underbrace{r^2 \frac{R''(r)}{R(r)} + \frac{2r R'(r)}{R(r)}}_{= k \quad \text{(constant)}} = -\frac{1}{\sin\varphi} \frac{\partial}{\partial\varphi}\left(\sin\varphi \cdot \Phi'(\varphi)\right)$$

Study of the radial equation (Euler's ODE)

$$r^2 R''(r) + 2r R'(r) - k R(r) = 0$$

Ansatz: $R(r) = r^\alpha$ w/ α to be determined

$$\alpha(\alpha-1) r^\alpha + 2\alpha r^\alpha - k r^\alpha = 0$$

$$(\alpha^2 + \alpha - k) r^\alpha = 0$$

Ansatz gives non-trivial solutions if $\boxed{\alpha^2 + \alpha - k = 0}$

$$\alpha_{1,2} = \frac{-1 \pm \sqrt{1+4k}}{2}$$

Claim: to have bounded and smooth solutions (as $r \to 0^+$) one needs $\sqrt{1+4k}$ to be an integer

$\iff$ $1+4k$ is a square

$\iff$ $\boxed{k = n(n+1)}$ for some integer n
(...) $(n = 0, 1, 2, 3, \dots)$

$$\alpha_{1,2} = \frac{-1 \pm \sqrt{1+4n+4n^2}}{2} = \frac{-1 \pm (2n+1)}{2} \begin{cases} = -n-1 \\ = +n \end{cases}$$

so $R(r) = a r^n + \underbrace{b r^{-n-1}}_{\text{blow-up at the origin}}$ (for some $a, b \in \mathbb{R}$)

$\Rightarrow \boxed{R(r) = a r^n}$ $n = 0, 1, 2, 3, \dots$

Angular equation (Legendre's equation)

$$\begin{cases} \dfrac{\partial}{\partial \varphi}\left(\sin\varphi \cdot \Phi'(\varphi)\right) + k \sin\varphi \cdot \Phi = 0 \\ k = n(n+1) \qquad n = 0, 1, 2, 3, \ldots \end{cases}$$

Solutions: the only bounded solutions to the equation above, for $k = n(n+1)$ and $n = 0, 1, 2, 3, \ldots$ are given by (Legendre polynomials)

$$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

evaluated at $\boxed{x = \cos\varphi}$.

Some low order examples:

$n = 0$	$P_0(x) = 1$	$P_0(\cos\varphi) = 1$
$n = 1$	$P_1(x) = x$	$P_1(\cos\varphi) = \cos\varphi$
$n = 2$	$P_2(x) = \frac{1}{2}(3x^2 - 1)$	$P_2(\cos\varphi) = \frac{1}{2}(3\cos^2\varphi - 1)$
...	...	...

Summary: our Ansatz ⊕ separation of variables led us to look for solutions of the form

$$\boxed{u(r, \varphi) = \sum_{n \geq 0} a_n r^n P_n(\cos\varphi)}$$

and we need to determine $a_0, a_1, a_2, \ldots$ using the BC namely

$$(*) \qquad u(1, \varphi) = g(\varphi)$$

given (known) function

General procedure

we must use the fact that the Legendre polynomials are orthogonal on $[-1,1]$, and in fact

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2m+1} & \text{if } n = m \end{cases}$$

note that takes the equivalent form

$$\int_0^{\pi} P_n(\cos\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2m+1} & \text{if } n = m \end{cases}$$

" $\cos\varphi = x$ "

So: $\quad u(r,\varphi) = \sum_{n\geq 0} a_n r^n P_n(\cos\varphi)$

$$\hookrightarrow u(1,\varphi) = \sum_{n\geq 0} a_n P_n(\cos\varphi) \overset{!}{=} g(\varphi)$$

We multiply by $P_m(\cos\varphi)\sin\varphi$ and integrate $\int_0^{\pi}$

$$\int_0^{\pi} \sum_{n\geq 0} a_n P_n(\cos\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi = \int_0^{\pi} g(\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi$$

$$\underbrace{\sum_{n\geq 0} \int_0^{\pi} a_n P_n(\cos\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi}_{a_m \frac{2}{2m+1}} = \int_0^{\pi} g(\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi$$

$$\rightsquigarrow \boxed{a_m = \frac{2m+1}{2} \int_0^{\pi} g(\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi}$$

Treatment of the special case: $\begin{cases} \Delta u = 0 & B_1 \\ u = \underbrace{1-\cos(2\varphi)}_{g(\varphi)} & \partial B_1 \end{cases}$

Given the discussion above, one needs to write $g(\varphi)$ in terms of Legendre polynomials, i.e. write

$$g(\varphi) = \sum_{n \geq 0} a_n P_n(\cos\varphi)$$

Approach 1: blindly apply formula above $\boxed{}$

Approach 2: $\cos 2\varphi = 2\cos^2\varphi - 1$

$$g(\varphi) = 1 - \cos 2\varphi = 2 - 2\cos^2\varphi$$

recall $\begin{cases} P_0(\cos\varphi) = 1 \\ P_1(\cos\varphi) = \cos\varphi \end{cases}$

$$P_2(\cos\varphi) = \frac{1}{2}(3\cos^2\varphi - 1)$$

thus

$$g(\varphi) = \underbrace{\frac{4}{3} P_0(\cos\varphi)}_{\frac{4}{3}\cdot 1} - \frac{4}{3} P_2(\cos\varphi)$$

$$- \frac{4}{3}\cdot\frac{3}{2}\cos^2\varphi + \frac{4}{3}\cdot\frac{1}{2}$$

hence

$$u(1,\varphi) = \sum a_n P_n(\cos\varphi) = \frac{4}{3} P_0(\cos\varphi) - \frac{4}{3} P_2(\cos\varphi)$$

$$\Longrightarrow \quad a_0 = 4/3, \quad a_2 = -4/3, \quad a_n = 0 \quad \forall n \in \mathbb{N} \setminus \{0,2\}$$

$$\Longrightarrow \quad u(r,\varphi) = \frac{4}{3} P_0(\cos\varphi) - \frac{4}{3} r^2 P_2(\cos\varphi)$$

$$= \frac{4}{3} - \frac{2}{3} r^2 (3\cos^2\varphi - 1).$$