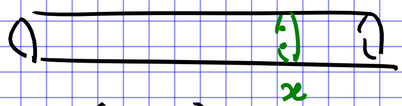
Diffusion-type equations (HEAT equation)

0 Physical experiment

→ metal rod (e.g. steel, copper, ..) length L

homogeneous, thin → temperature $u(x,t)$



→ initial state $u(x,0) = u_0(x)$ known (e.g. 10°C)

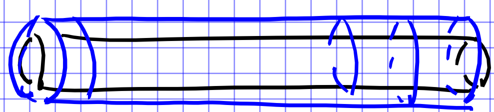
→ boundary conditions $u(0,t) = T_1$ (e.g. 0°C)

$u(L,t) = T_2$ (e.g. 50°C)

(rmk. T_1, T_2 are not needed to be constant in time

i.e. it could be $T_1(t)$, $T_2(t)$)

we let the system evolve, with the side surface thermically insulated.



Q. determine $u(x,t)$, i.e. temperature at all points, and for all times. For instance: what will be the "asymptotic state" of the system?

1 To answer this question, we first need to set up the problem mathematically, by means of a PDE. Fourier's law tells us that the evolution of this system obeys the equation

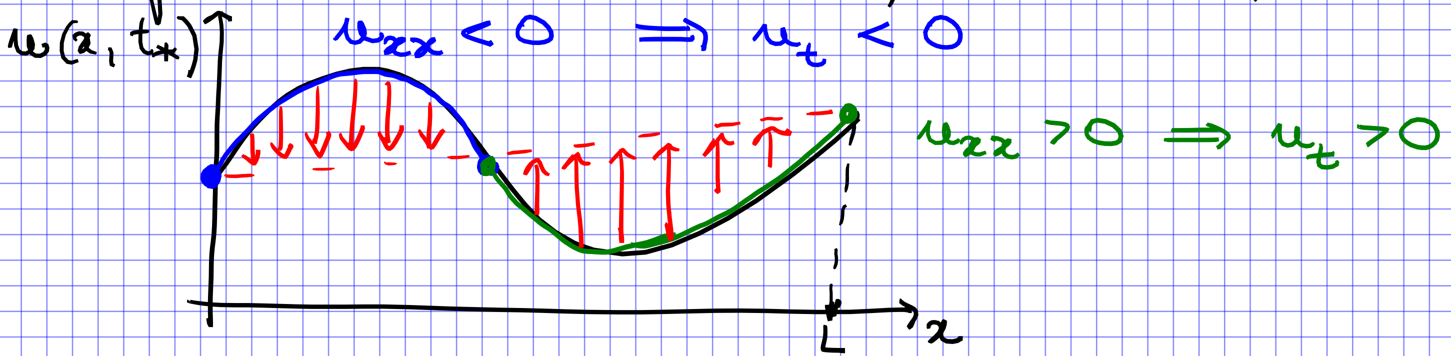
$$u_t = \alpha\, u_{xx}$$

which we need to couple w/ initial and boundary conditions to get an Initial Boundary Value Problem (IBVP)

$$(*)\quad \begin{cases} u_t = \alpha u_{xx} \\ u(x,0) = u_0(x) \\ u(0,t) = T_1(t) \\ u(L,t) = T_2(t) \end{cases}$$

We'll see (next lectures) that $(*)$ is a well-posed problem: there exists a unique solution $u(x,t)$ describing the temperature of the rod.

Heuristics: let's consider an hypothetical plot of the temperature at some time t_*, $u(x,t_*)$



We'll see that the asymptotic state of $u(x,t)$ is the linear interpolation of T_1, T_2 (at least in the simpler case when such boundary conditions are constant in time ...)

(in particular this result, i.e. the asymptotic state, does not depend on the initial condition u_0)

Rmk. if we allow for an internal heat source, we'll have to add an extra term to the equation above

$$u_t = \alpha u_{xx} + f(x,t)$$

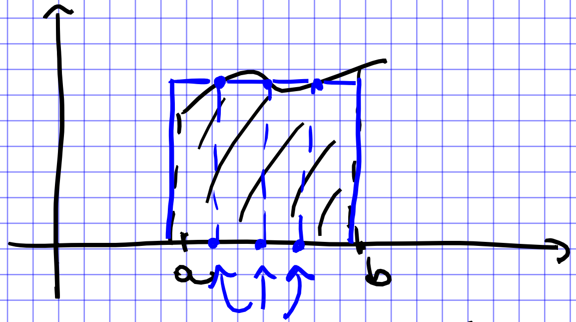
i.e. we have an inhomogeneous heat equation. ■

(2) "Deriving" the heat equation

mathematical review: fundamental thm of Calculus

$f \in C^0([a,b], \mathbb{R})$

$\exists\, c \in [a,b]$ such that $\dfrac{\int_a^b f(s)\,ds}{b-a} = f(c)$



In particular,

$$\lim_{b \to a^+} \frac{\int_a^b f(s)\,ds}{b-a} = f(a)$$

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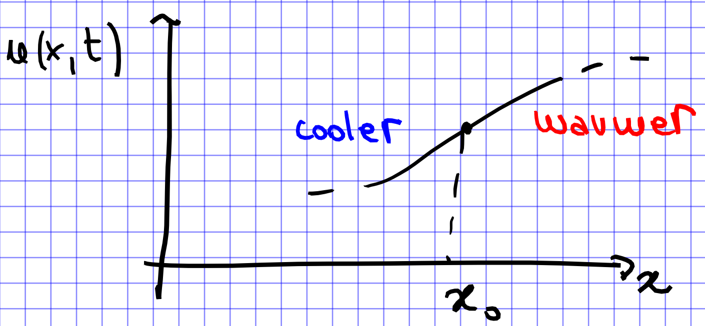
We'll derive the heat equation from two basic laws in classical thermodynamics:

*) Fourier's law

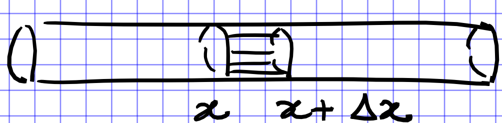


Flux of heat crossing x_0 (from left to right) $= -\,\kappa\, A\, u_x$

thermal conductivity → κ; area of cross-sec → A

physically: energy per unit of time so measured in W

**) Conservation law



$$\underbrace{\text{Net change of heat content in } [x, x+\Delta x]}_{(1)} = \underbrace{\text{Net flux of heat across the boundries}}_{(2)} + \underbrace{\text{Total heat generated inside } [x, \Delta x]}_{(3)}$$

$$(1) = \frac{\partial}{\partial t} \int_x^{x+\Delta x} c\,\rho\, A\, u(s,t)\,ds$$

c: thermal capacity of the rod; ρ: density; A: area of the cross-section

$$(2) \underset{\text{Fourier law}}{=} -KA\,u_x(x,t) - \left(-KA\,u_x(x+\Delta x, t)\right)$$

$$(3) = \int_x^{x+\Delta x} A \underbrace{f(s,t)}_{} ds$$

calories generated per unit of volume (description of the heat sources)

Let's combine them, and plug-in the balance equation:

$$\frac{\partial}{\partial t}\int_x^{x+\Delta x} c\rho\, A u(s,t)\,ds = KA\left[u_x(x+\Delta x,t) - u_x(x,t)\right] + \int_x^{x+\Delta x} A\, f(s,t)\,ds$$

$$\int_x^{x+\Delta x} c\rho\, \cancel{A}\, u_t(s,t)\,ds = K\cancel{A}\left[u_x(x+\Delta x,t) - u_x(x,t)\right] + \int_x^{x+\Delta x} \cancel{A}\, f(s,t)\,ds$$

divide by $c\rho\,\Delta x$ and then let $\Delta x \to 0$

$$\lim_{\Delta x\to 0} \frac{\int_x^{x+\Delta x} u_t(s,t)\,ds}{\Delta x} = \lim_{\Delta x\to 0} \frac{K}{c\rho}\left[\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right] + \lim_{\Delta x\to 0} \frac{1}{c\rho}\frac{\int_x^{x+\Delta x} f(s,t)\,ds}{\Delta x}$$

$$u_t(x,t) \qquad \frac{K}{c\rho} u_{xx}(x,t) \qquad \frac{1}{c\rho} f(x,t)$$

$$\rightsquigarrow \quad u_t(x,t) = \alpha\, u_{xx}(x,t) + F(x,t)$$

where we have set $\alpha := \dfrac{K}{c\rho}$, $F(x,t) = \dfrac{1}{c\rho} f(x,t)$

3 Variations on the theme above:

3.1 Other BC

hom. case

$$\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases}$$

imposed temperature $\quad T_1 = T_2 = 0$

Dirichlet

$$\begin{cases} u_x(0,t) = h_1 \\ u_x(L,t) = h_2 \end{cases}$$

imposed flux

$h_1 = h_2 = 0$

Neumann

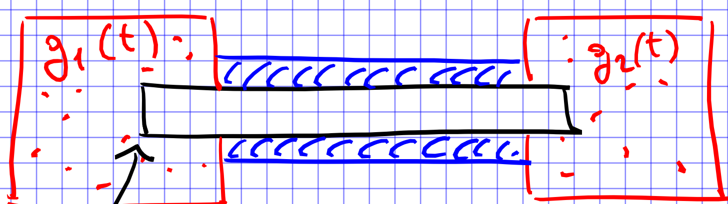
Most general version:

$$\begin{cases} u_x(0,t) + \lambda u(0,t) = g_1(t) \\ u_x(L,t) + \lambda u(L,t) = g_2(t) \end{cases}$$

$g_1 = g_2 = 0$

Robin

Comment on ↑ BC:



flux at this interface

by Fourier $= k u_x$

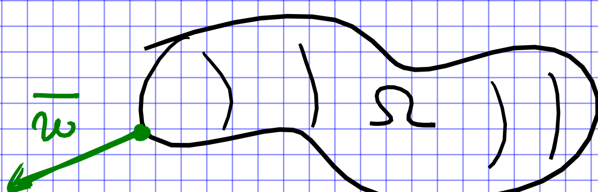
$= h\,[u - g_1]$

Newton's law

$\rightsquigarrow \quad u_x - \frac{h}{k} u = - \frac{h}{k} g_1$

3.2 Multi-dimensional problem (typically 2D or 3D)

Let's consider a 3D body (homogeneous, conductive etc...)



heat equation

$u_t = \alpha \Delta u$

$u\,(\underbrace{x, y, z}_{\text{3 spatial variables}}, t)$

here $\Delta u := u_{xx} + u_{yy} + u_{zz}$

initial condition

$u(t = t_0) = u_0$

given function

boundary conditions

- imposed temperature $\quad u\big|_{\partial\Omega} = T$ — given function

- imposed flux $\quad \dfrac{\partial u}{\partial \bar{n}}\Big|_{\partial\Omega} = h$ — given function

Achtung!

- $\dfrac{\partial u}{\partial \bar{n}} = \nabla u \cdot \bar{n}$

- for radially symm problems $\quad \dfrac{\partial u}{\partial \bar{n}} = u_r$ — radial derivative

Mixed-type conditions $\quad \dfrac{\partial u}{\partial \bar{n}} + \lambda u = g$ — given function