SEPARATION OF VARIABLES (applied to heat equation)

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{(PDE)} \\ u(0,t) = 0 & \multirow{2}{*}{\} BC} \\ u(1,t) = 0 & \\ u(x,0) = \phi(x) & \} \text{ IC} \end{cases}$$

$\phi(x)$ ← known function

goal: solve this IBVP for $u(x,t)$

We'll solve it by separation of variables, i.e. we make an Ansatz and start by looking for solutions of the form

$$u(x,t) = X(x)\,T(t)$$

here $0 \le x \le 1$, $t \ge 0$.

Step 1: solutions to the sole PDE

$$X(x)\,T'(t) = \alpha^2 X''(x)\,T(t)$$

$$\hookrightarrow \underbrace{\frac{X''(x)}{X(x)}}_{\text{does not depend on } t} = \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{does not depend on } x} \rightsquigarrow$$

either side is actually a constant i.e.

$$\begin{cases} X''/X = \kappa \\ T'/\alpha^2 T = \kappa \end{cases} \quad \text{for some } \kappa \in \mathbb{R}$$

solutions (review of ODE theory):

- $\kappa = 0 \longrightarrow X(x) = A + Bx$ for some $A, B \in \mathbb{R}$
- $\kappa > 0$ ($\kappa = \lambda^2$) $\longrightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ " " "
- $\kappa < 0$ ($\kappa = -\lambda^2$) $\longrightarrow X(x) = A\sin(\lambda x) + B\cos(\lambda x)$ " " "

For the 2nd equation, we (always) get $T(t) = Ce^{\kappa\alpha^2 t}$.

Step 2: impose the boundary conditions

$$u(0,t) = 0 = X(0)T(t) \overset{\text{in order to find non-trivial sol.}}{\Longrightarrow} X(0)=0$$

$$u(1,t) = 0 = X(1)T(t) \Longrightarrow X(1)=0$$

$(\forall t \geq 0)$

So, we get the problem (to be solved for $X(x)$)

$$\begin{cases} X'' = kX \\ X(0)=0 \qquad X(1)=0 \end{cases}$$

Rmk. the only case where we get for non-trivial sols. is $k<0$ (i.e. $k=-\lambda^2$), so by discussion above

$X(x) = A\sin(\lambda x) + B\cos(\lambda x)$ and the BC impose

$$\begin{cases} X(0) = B = 0 \\ X(1) = A\sin(\lambda) = 0 \end{cases} \Longrightarrow \begin{matrix} B=0 \\ \lambda = n\pi \qquad n=1,2,3,\dots \end{matrix}$$

so we find a sequence of solutions $\boxed{X_n(x) = A_n \sin(n\pi x)}$

and, associated to X_n, so for $k=-\lambda^2=-(n\pi)^2$ we get $T_n(t) = e^{-(n\pi)^2\alpha^2 t}$

So we have for PDE $\oplus$ BC a sequence of solutions

$$u_n(x,t) = X_n(x)T_n(t) = A_n \sin(n\pi x)\, e^{-(n\pi)^2\alpha^2 t}$$

thus by linearity ($\leftrightarrow$ superposition principle) the most general sol. for PDE $\oplus$ BC

$$u(x,t) = \sum_{n\geq 1} u_n(x,t) \qquad \text{i.e.}$$

$$u(x,t) = \sum_{n\geq 1} A_n \sin(n\pi x)\, e^{-(n\pi)^2\alpha^2 t}$$

Step 3: impose initial condition $u(x,0) = \phi(x)$

The key question: can we actually choose (or: uniquely determine) $A_1, A_2, A_3, \dots$ so that

$$u(x,0) = \sum_{n\geq 1} A_n \sin(n\pi x) \overset{(\#)}{=} \phi(x)$$

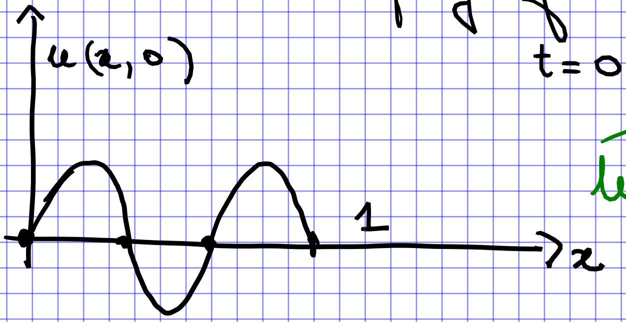
Toy model: $\phi(x) = 5\sin(3\pi x)$ in this case it's clear that

I can write: $A_1 = A_2 = 0 \quad A_3 = 5 \quad A_n = 0$ for $n \geq 4$

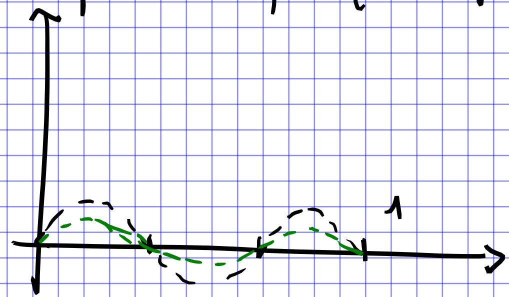
$\hookrightarrow$ the solution then becomes

$$\boxed{u(x,t) = 5 \sin(3\pi x)\, e^{-\alpha^2 (3\pi)^2 t}}$$

comment: damping of the initial temperature profile $\phi(x)$



let time evolve



<u>ANSWER</u> (in general): YES, provided the datum ϕ is "good enough" (i.e. regular). The theory of Fourier series ensures that indeed there exist unique coefficients so that (#) holds. Before developing the theory, let's compute $\{A_n\}$ brute force. Suppose (#) holds, multiply it by $\sin(m\pi x)$

free parameter we introduce $m = 1, 2, 3, \ldots$

and integrate

$$\int_0^1 \sum_{n \geq 1} A_n \sin(n\pi x) \sin(m\pi x)\, dx = \int_0^1 \phi(x) \sin(m\pi x)\, dx$$

<u>Fact</u>

$$\int_0^1 \sin(n\pi x) \sin(m\pi x)\, dx = \begin{cases} 1/2 & \text{if } n = m \\ 0 & \text{else} \end{cases}$$

$$\hookrightarrow \quad A_m \cdot \frac{1}{2} = \int_0^1 \phi(x) \sin(m\pi x)\, dx$$

$$\hookrightarrow \quad \boxed{A_m = 2 \int_0^1 \phi(x) \sin(m\pi x)\, dx}$$

Final solution to our problem is then

$$u(x,t) = \sum_{n \geq 1} \left(2 \int_0^1 \phi(s) \sin(n\pi s)\, ds \right) \sin(n\pi x)\, e^{-\alpha^2 (n\pi)^2 t}$$

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FOURIER SERIES (real case, i.e. over $\mathbb{R}$)

⓪ Review on orthonormal bases

in $\mathbb{R}^n$ w/ standard dot product $\langle x, y\rangle = x_1 y_1 + \dots + x_n y_n$
we'll say that

ⓐ $\{v_1, \dots, v_n\}$ is an orthogonal basis if (it is a basis)
and
$$\langle v_i, v_j \rangle = 0 \quad \text{if } i \neq j$$

ⓑ $\{v_1, \dots, v_n\}$ is an orthonormal basis if it is an orthogonal basis consisting of unit-length vectors, i.e. a basis satisfying
$$\langle v_i, v_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Fact: if $\{v_1, \dots, v_n\}$ is an orthogonal basis then any $v \in \mathbb{R}^n$ can be written as
$$v = \sum_{i=1}^{n} \frac{\langle v, v_i\rangle}{\langle v_i, v_i\rangle} v_i$$

Special case: if $\{v_1, \dots, v_n\}$ is orthonormal then
in fact $\quad v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$

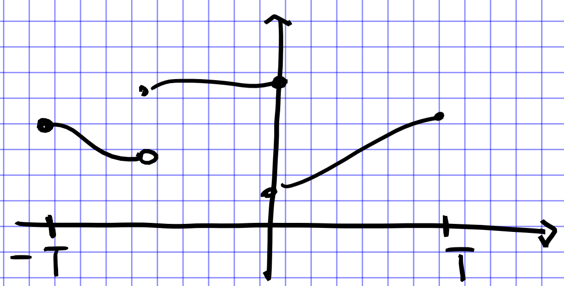
Example: in $\mathbb{R}^3$ the standard Euclidean basis
$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
is an orthonormal basis; when we write $x \in \mathbb{R}^3$ in coordinates
$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{then we mean} \quad x = x_1 e_1 + x_2 e_2 + x_3 e_3$$
and indeed
$$x_i = \frac{\langle x, e_i \rangle}{\langle e_i, e_i \rangle} = \langle x, e_i \rangle$$

① Setup for Fourier series

Given $T>0$ we let

$$X = \{ f : (-T,T) \longrightarrow \mathbb{R} \ \text{ that are piecewise } C^1 \}$$

$\mathbb{R}^3$



definition:

$f \in X$ iff $\exists$ partition $\{t_0 = -T < t_1 < \dots < t_k = T\}$ and $C = C(f)$

such that $f|_{(t_i, t_{i+1})}$ is C^1, and $|f(t)| + |f'(t)| \leq C$ $\forall\, t \in (t_i, t_{i+1})$ $\forall\, i \in \{0, 1, \dots, k-1\}$

Fact (problem 3.2): if $f \in X$ then $\forall\, i \in \{0, 1, \dots, k-1\}$

$\exists$ left limit, i.e. $f_-(t_i) := \lim_{t \to t_i^-} f(t)$

$\exists$ right limit, i.e. $f_+(t_i) := \lim_{t \to t_i^+} f(t)$

- Dot product on X:

$$\langle f, g \rangle := \int_{-T}^{T} f(s)\, g(s)\, ds$$

dot product in $\mathbb{R}^3$

- An Hilbertian basis for $(X, \langle\,,\,\rangle)$ is given by

$$\left\{ \begin{aligned} e_0 &= \frac{1}{\sqrt{2T}} \\ e_n &= \frac{1}{\sqrt{T}} \cos\left(\frac{n\pi t}{T}\right) \\ f_n &= \frac{1}{\sqrt{T}} \sin\left(\frac{n\pi t}{T}\right) \end{aligned} \right\} \quad n \in \{1, 2, 3, \dots\}$$

standard Euclidean basis in $\mathbb{R}^3$

check: this is an orthonormal family

- The (real) <u>Fourier series</u> of $f \in X$ is given by

$$S_f = A_0 e_0 + \sum_{n \geq 1} (A_n e_n + B_n f_n)$$

where
$$A_0 = \langle f, e_0 \rangle$$
$$A_n = \langle f, e_n \rangle$$
$$B_n = \langle f, f_n \rangle$$

Equivalently:
$$S_f(t) = \frac{a_0}{2} + \sum_{n \geq 1} \left[a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right]$$

where
$$a_n = \frac{1}{T} \int_{-T}^{T} f(s) \cos\left(\frac{n\pi s}{T}\right) ds, \quad b_n = \frac{1}{T} \int_{-T}^{T} f(s) \sin\left(\frac{n\pi s}{T}\right) ds$$

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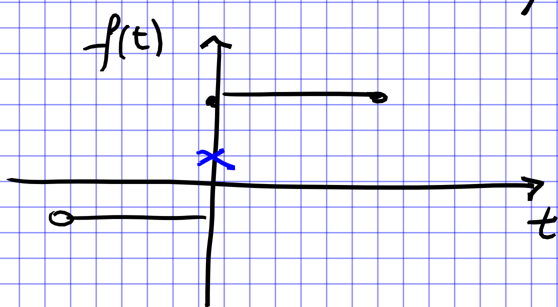
<u>Convergence</u> (a.k.a.: what is the relation between a function $f \in X$, and its Fourier series?)

① $S_f(t) \xrightarrow[\text{(pointwise)}]{} f(t)$ whenever f is continuous (i.e. if f is continuous at t)

② $S_f(t) \xrightarrow[\text{(pointwise)}]{} \dfrac{f_+(t) + f_-(t)}{2}$ otherwise (i.e. at $t = t_i$)

<u>Example</u>: $T = 1$

$$f(t) = \begin{cases} 1 & t \in [0, 1) \\ -1/2 & t \in (-1, 0) \end{cases}$$



for $t = 0$ we have $S_f(0) \longrightarrow \dfrac{1 + (-1/2)}{2} = 1/4$