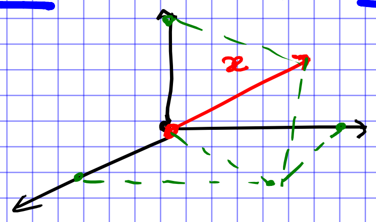
LECTURE 4

14/10/2021

Review on (real) Fourier series

⓪ **Idea** :



$\mathbb{R}^3$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

equivalently $x = x_1 e_1 + x_2 e_2 + x_3 e_3$

Formulae: $f \in X \rightsquigarrow$ decompose it in "fundamental harmonics"

$$S_f(t) = \frac{a_0}{2} + \sum_{n \geq 1} \left[a_n \cos\left(\frac{\pi n t}{T}\right) + b_n \sin\left(\frac{\pi n t}{T}\right) \right]$$

$$a_n = \frac{1}{T} \int_{-T}^{T} f(s) \cos\left(\frac{\pi n s}{T}\right) ds \qquad b_n = \frac{1}{T} \int_{-T}^{T} f(s) \sin\left(\frac{\pi n s}{T}\right) ds$$

① Even and Odd functions

for $f \in X$ (in particular $f : (-T, T) \to \mathbb{R}$) we say:

(i) f is EVEN if $f(-t) = f(t)$

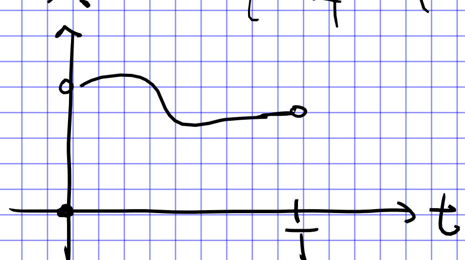
$\longrightarrow$ in this case $b_n = 0 \ \forall n \geq 1$ (no sines in the Fourier s)

(ii) f is ODD if $f(-t) = -f(t)$

$\longrightarrow$ in this case $a_n = 0 \ \forall n \geq 0$ (no cosines in Fourier s.)

② Functions only defined "on one side"

$$\tilde{X} = \{ \tilde{f} : (0, T) \to \mathbb{R} \ \text{piecewise } C^1 \}$$



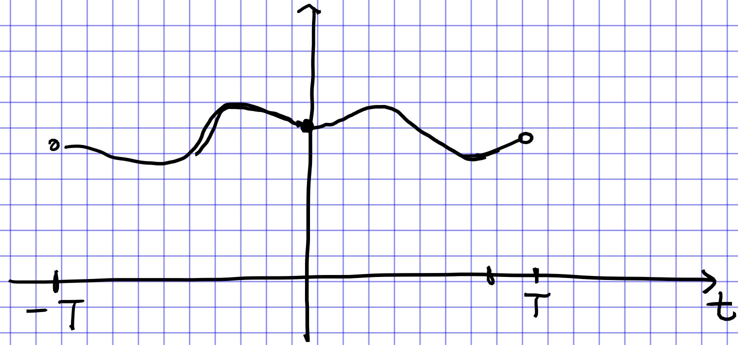
notation: $\tilde{f}_0 = \lim_{t \to 0^+} \tilde{f}(t)$

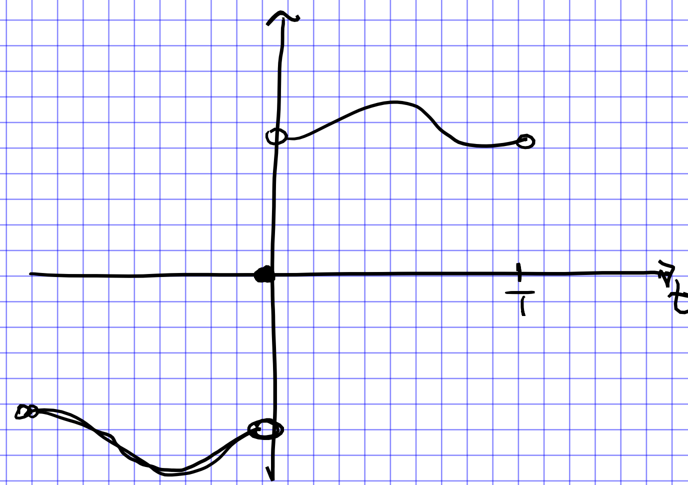
Any $\tilde{f} \in \tilde{X}$ may give rise to:

- an EVEN extension, by letting $f_E(t) = \begin{cases} \tilde{f}(t) & 0 < t < T \\ \tilde{f}(-t) & -T < t < 0 \\ \tilde{f}_0 & t = 0 \end{cases}$

  $(f_E \in X)$

- an ODD extension, by letting $f_O(t) = \begin{cases} \tilde{f}(t) & 0 < t < T \\ -\tilde{f}(-t) & -T < t < 0 \\ 0 & t = 0 \end{cases}$





Hence:

- f_E has a Fourier series only in terms of $\cos\left(\frac{\pi n t}{T}\right)$ $(n \geq 0)$

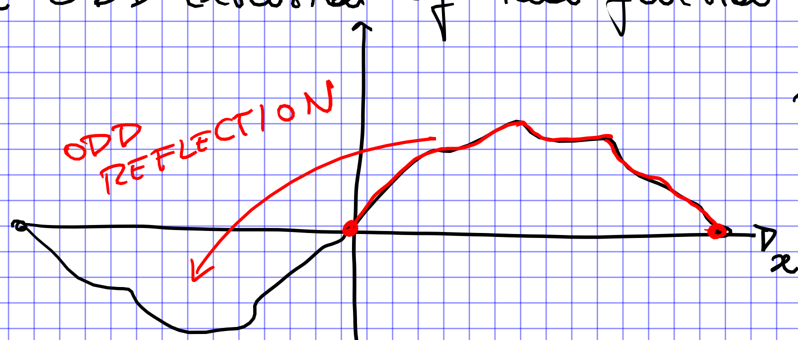
- f_O has a Fourier series only in terms of $\sin\left(\frac{\pi n t}{T}\right)$ $(n \geq 1)$

if I restrict back to $(0,T)$ i.e. look at $0<t<T$

$$f(t) = f_E(t) = \sum_{n\geq 0} A_n e_n$$

$$f(t) = f_O(t) = \sum_{n\geq 1} B_n f_n$$

So any $\tilde{f} \in \tilde{X}$ has, at least a priori, two different Fourier expansions. In separation of variables last time (IBVP for heat equation) we implicitly employed the $\sin(\cdot)$ expansion of $\phi(x)$, i.e. we were really considering the Fourier series of the ODD extension of that function ϕ.



$u(x,0) = \phi(x)$

$0 < x < L$

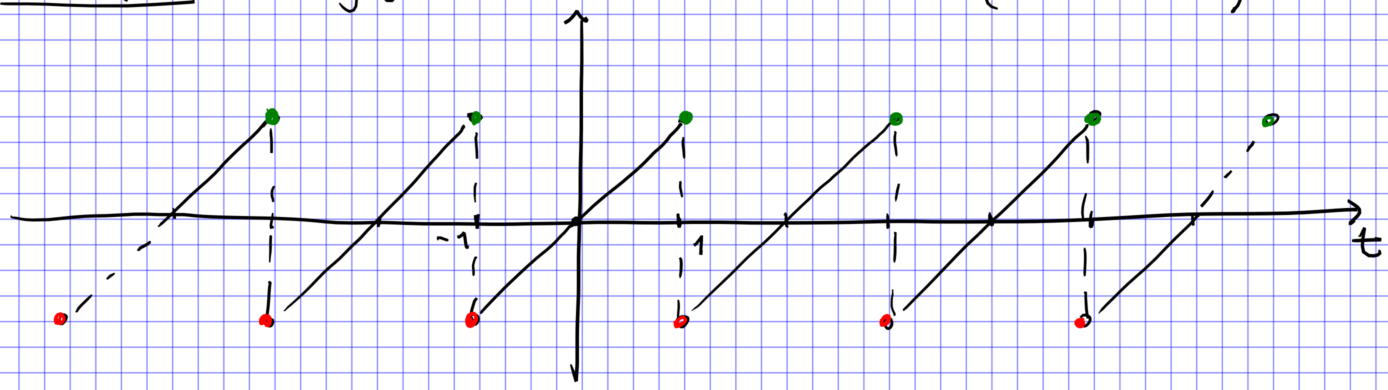
③ Periodic Functions and periodic extensions

$f : (-T,T) \longrightarrow \mathbb{R}$ $\rightsquigarrow$ $S_f : (-T,T) \longrightarrow \mathbb{R}$

$f : \mathbb{R} \longrightarrow \mathbb{R}$ (by periodic extension, period $= 2T$)

$S_f : \mathbb{R} \longrightarrow \mathbb{R}$ (by periodic extension, period $= 2T$)

Example: $f(t) = t \quad -1 < t < 1 \quad$ (i.e. $T = 1$)



- extension by left-continuity
- extension by right-continuity

} these are consistent if and only if $f(-T) = f(T)$

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Complex Fourier series

⓪ Review on (complex) orthonormal bases

in $\mathbb{C}^n$, standard hermitian product

$$(u, v \in \mathbb{C}^n) \qquad \langle u, v \rangle = \sum_{i=1}^{n} u_i \overline{v_i}$$

we can define a norm (measure length, angles...)
by letting $\|u\| := \sqrt{\langle u, u \rangle}$; we'll say that

ⓐ $\{u^{(1)}, \dots, u^{(n)}\}$ is an orthogonal basis if (it is a basis)
and $\langle u^{(i)}, u^{(j)} \rangle = 0$ if $i \neq j$

ⓑ $\{u^{(1)}, \dots, u^{(n)}\}$ is an orthonormal basis if it is an orthogonal basis consisting of unit-length vectors, i.e. a basis satisfying

$$\langle u^{(i)}, u^{(j)} \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Fact: if $\{u^{(1)}, \dots, u^{(n)}\}$ is an orthogonal basis
then any $u \in \mathbb{C}^n$ can be written as

$$u = \sum_{i=1}^{n} \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} u_i \qquad (*)$$

Example: in $\mathbb{C}^3$ the standard hermitian basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is an orthonormal basis (over $\mathbb{C}$); when we write $u \in \mathbb{C}^3$ in coordinates $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $u_1, u_2, u_3 \in \mathbb{C}$ we mean (*)

i.e. $u = u_1 e_1 + u_2 e_2 + u_3 e_3$.

1 Setup for complex Fourier series

- Given $T > 0$ we let

$$X_{\mathbb{C}} := \{ f : (-T, T) \longrightarrow \mathbb{C} \text{ that are piecewise } C^1 \}$$

- Hermitian product on $X_{\mathbb{C}}$ $\quad \langle f, g \rangle := \int_{-T}^{T} f(s)\, \overline{g(s)}\, ds$
- An Hilbertian basis for $(X_{\mathbb{C}}, \langle\ ,\ \rangle)$, in part. an orthonormal basis for is

$$\left\{ e_n = \frac{e^{i\left(\frac{\pi n t}{T}\right)}}{\sqrt{2T}} \right\} \text{ where } \boxed{n \in \mathbb{Z}}$$

- The complex Fourier series of $f \in X_{\mathbb{C}}$ is given by

$$S_f^{\mathbb{C}} = \sum_{n \in \mathbb{Z}} C_n e_n$$

$$C_n = \langle f, e_n \rangle$$

Equivalently
$$S_f^{\mathbb{C}}(t) := \sum_{n \in \mathbb{Z}} c_n e^{i\left(\frac{\pi n t}{T}\right)}$$

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(s)\, e^{-i\left(\frac{\pi n s}{T}\right)} ds$$

2 Link between real and complex form:

if $x \in \mathbb{R}$, $\quad e^{ix} = \cos(x) + i \sin(x)$

this formula allows to get the following equations

Let $f \in X \subset X_{\mathbb{C}}$, then:

$$\begin{cases} c_0 = \dfrac{a_0}{2} & \\ c_n = \dfrac{a_n - i b_n}{2} & (n = 1, 2, 3, \dots) \\ c_{-n} = \dfrac{a_n + i b_n}{2} & (n = 1, 2, 3, \dots) \end{cases} \qquad \begin{cases} a_0 = 2c_0 \\ a_n = c_n + c_{-n} \\ b_n = \dfrac{c_{-n} - c_n}{i} \end{cases}$$

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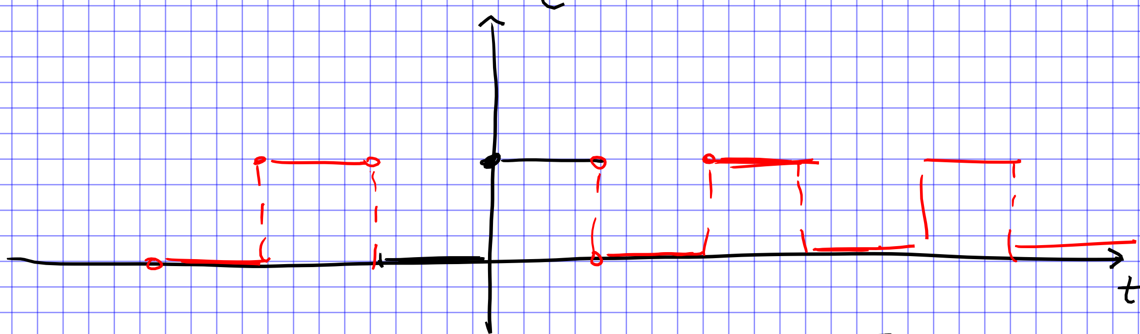
<u>Example 1</u> : square wave

$T = \pi$ $\qquad f(t) = \begin{cases} 1 & 0 \le t < \pi \\ 0 & -\pi < t < 0 \end{cases}$



$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)\, e^{-ins}\, ds = \frac{1}{2\pi} \int_0^{\pi} e^{-ins} ds$$

$n = 0 \longrightarrow \quad c_0 = 1/2$

$n \neq 0 \longrightarrow \quad c_n = \dfrac{1}{2\pi} \left[-\dfrac{1}{in} e^{-ins} \right]_{s=0}^{s=\pi}$

$$= \left(\frac{1}{2\pi i n} \right) \left[1 - e^{-in\pi} \right]$$

$$= \begin{cases} \dfrac{1}{\pi n i} & \text{if } n \text{ ODD} \\ 0 & \text{if } n \text{ even} \end{cases}$$

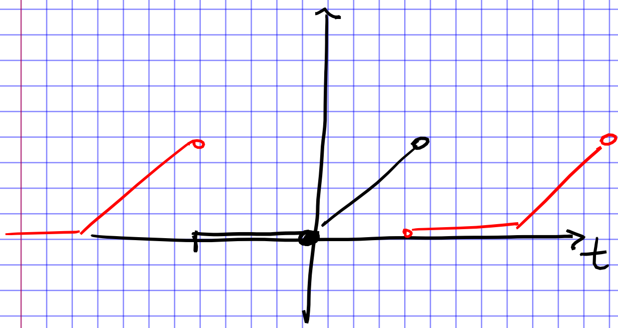
$$S_f^{\mathbb{C}} = \frac{1}{2} + \sum_{\substack{n \in \mathbb{N} \\ n \text{ ODD}}} \frac{1}{\pi n i} \underbrace{\left(e^{int} - e^{i(-n)t} \right)}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{n \in \mathbb{N} \\ n \text{ ODD}}} \frac{\sin(nt)}{n} \longleftarrow \text{real Fourier series of } f$$

Example 2

$T = \pi$ $\qquad f(t) = \begin{cases} t & \text{if } 0 \leq t < \pi \\ 0 & \text{if } -\pi < t < 0 \end{cases}$



$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)\,ds = \frac{1}{2\pi} \int_0^{\pi} s\,ds = \frac{\pi}{4}$$

$$c_n = \begin{cases} -\dfrac{1}{2in} & n \text{ even} \\[2mm] \dfrac{1}{2in} - \dfrac{1}{\pi n^2} & n \text{ odd} \end{cases}$$

The convergence theorem ensure pointwise convergence at $t=0$

This means: $\quad 0 = f(0) = \frac{\pi}{4} + \sum_{n=1}^{\infty} (c_n + c_{-n})$

$$= \frac{\pi}{4} + \sum_{\substack{n=1 \\ \text{ODD}}}^{\infty} -\frac{2}{\pi n^2}$$

formula above

$$\Longleftrightarrow \quad \frac{\pi}{4} = \sum_{\substack{n=1 \\ \text{ODD}}}^{\infty} \frac{2}{\pi n^2}$$

$$\Longleftrightarrow \quad \boxed{\frac{\pi^2}{8} = \sum_{\substack{n=1 \\ \text{ODD}}}^{\infty} \frac{1}{n^2}}$$

(Euler, Basel problem)