LECTURE 5 21/10/2021

Separation of variables - reloaded

Exercise 1: how to handle Robin BCs.

Solve, by sep. of variables, the IBVP

$0 < x < 1$

$t > 0$

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(0,t) = 0 \\ u_x(1,t) + h\,u(1,t) = 0 \\ u(x,0) = x \end{cases}$$

physical parameter > 0

Solution: the solution is given by an (infinite) sum of special solutions of the form $X(x)\,T(t)$.

So let's plug in this Ansatz and solve for X and T.

Step 1: $T'(t)\,X(x) = \alpha^2 X''(x)\,T(t)$

$\longrightarrow$ $$\underbrace{\frac{X''(x)}{X(x)}}_{\mu} = \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\mu} \qquad (\mu \in \mathbb{R})$$

$\longrightarrow$ both constant

We then look at the equation for $X(x)$ together w/ BCs.

$$\begin{cases} X(0)\,T(t) = 0 \\ X'(1)\,T(t) + h\,X(1)\,T(t) = 0 \end{cases}$$

$\rightsquigarrow$ non-trivial solutions

$$\begin{cases} X(0) = 0 \\ X'(1) + h\,X(1) = 0 \end{cases}$$

So we get for $X(x)$ the problem (*) $$\begin{cases} X'' = \mu X \\ X(0) = 0 \\ X'(1) + h\,X(1) = 0 \end{cases}$$

(This is a special case of a Sturm-Liouville pb.)

Step 2: discussing the separation constant.

Distinguish 3 cases based on $\operatorname{sign}(\mu)$:

- $\mu = 0 \implies X(x) = A + Bx \quad (\forall A, B \in \mathbb{R})$

$$X(0) = 0 = A \implies A = 0$$

$$X'(1) + h X(1) = B + hB = 0$$
$$= B(1+h) \implies B = 0$$

- $\mu > 0 \implies X(x) = A\cosh(\lambda x) + B\sinh(\lambda x) \quad (\forall A,B \in \mathbb{R})$

$(\mu = \lambda^2)$

$\lambda > 0$

$$X(0) = 0 = A \implies A = 0$$

$$X'(1) + h X(1) = 0 = \lambda B \cosh(\lambda) + hB\sinh(\lambda)$$
$$= B \lambda \cosh(\lambda) \underbrace{\left(1 + \frac{h \tanh(\lambda)}{\lambda}\right)}$$

positive if $\lambda > 0$

$$\implies B = 0$$

- $\mu < 0 \implies X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad (\forall A,B \in \mathbb{R})$

$\mu = -\lambda^2$

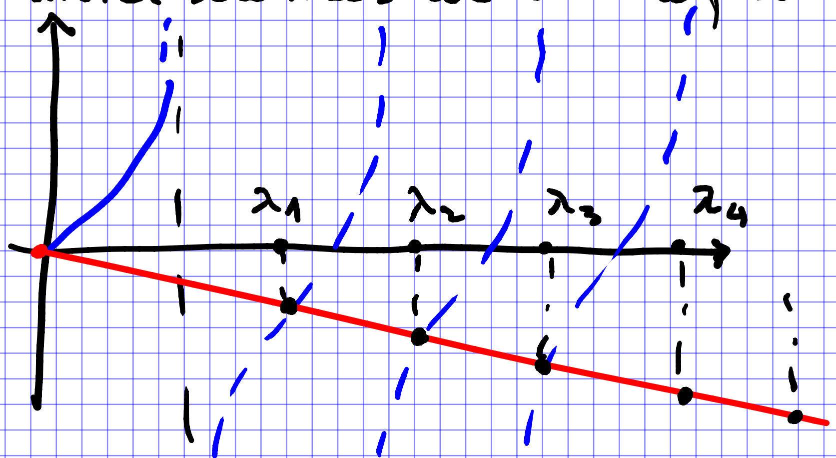
$(\lambda > 0)$

$$X(0) = 0 = A \implies A = 0$$

$$X'(1) + h X(1) = B\lambda\cos(\lambda) + hB\sin(\lambda)$$
$$= B\lambda\cos(\lambda)\left(1 + \frac{h\tan(\lambda)}{\lambda}\right)$$
$$= 0$$

So if we want non-trivial solutions we must impose

$$\tan\lambda = -\frac{1}{h}\lambda$$



we get a sequence $\{\lambda_n\}$ $n \geq 1$

So $X_n(x) = \sin(\lambda_n x)$ $n \geq 1$, and correspondingly

$T_n(t) = e^{-(\alpha\lambda_n)^2 t}$ and so our Ansatz is that

$$u(x,t) = \sum_{n \geq 1} a_n e^{-(\alpha\lambda_n)^2 t} \sin(\lambda_n x)$$

and one must determine $a_1, a_2, \dots$ so to match the I.C., i.e. $\sum_{n\geq 1} a_n \sin(\lambda_n x) = x$

<u>Step 3</u>: to find a_m apply "brute force trick"
multiply by $\sin(\lambda_m x)$ and integrate

$$\int_0^1 x \sin(\lambda_m x)\,dx = \sum_{n\geq 1} a_n \underbrace{\int_0^1 \sin(\lambda_n x)\sin(\lambda_m x)\,dx}$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{\lambda_m - \sin\lambda_m \cos\lambda_m}{2\lambda_m} & \text{if } n = m \end{cases}$$

$$\int_0^1 x \sin(\lambda_m x)\,dx = a_m \left(\frac{\lambda_m - \sin\lambda_m\cos\lambda_m}{2\lambda_m}\right)$$

$$a_m = \frac{2\lambda_m}{\lambda_m - \sin\lambda_m \cos\lambda_m}\int_0^1 x \sin(\lambda_m x)\,dx$$

Modulo discussing convergence of series $\sum_{n\geq 1} a_n \sin(\lambda_n x)$ (see below) the problem is solved.

<u>Rmk.</u> 1 in general, if we start from

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

one can argue as above, w/ (*) is substituted by the Sturm-Liouville pb.

$$\begin{cases} X'' = \mu X \\ \alpha_1 X'(0) + \beta_1 X(0) = 0 \quad k_1(t) \\ \alpha_2 X'(1) + \beta_2 X(1) = 0 \quad k_2(t) \end{cases}$$

eigenvalues of $\frac{d^2}{dx^2}$ under oblique BCs

by general "abstract" results there always exist sequences of $\begin{cases} \text{eigenvalues} & \lambda_1 \le \lambda_2 \le \lambda_3 \ldots \\ \text{eigenfunctions} & X_1 \quad X_2 \quad X_3 \ldots \end{cases}$

w/ • $\displaystyle\int_0^1 X_i(x)\, X_j(x)\, dx = 0$ if $i \neq j$

•• (pointwise) convergence theorems hold true

(2) Standard trick to reduce to homogeneous BCs "by subtracting an affine function". Discussed in detail in Farlow's Lesson 6, and problem 5.1.

Exercise 2 : solve inhomogeneous PDEs

$0 < x < 1$

$t > 0$

$$\begin{cases} u_t - \alpha^2 u_{xx} = \boxed{f(x,t)} \\ \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Step 1 (usual story ...) get to SL problem

$$\begin{cases} X'' = \mu X \\ \alpha_1 X'(0) + \beta_1 X(0) = 0 \\ \alpha_2 X'(1) + \beta_2 X(1) = 0 \end{cases}$$

step 2 ($\mu = -\lambda^2$)

$\longrightarrow$

(theory)

$\lambda_1 \le \lambda_2 \le \ldots$

$X_1 \le X_2 \le \ldots$

these are not explicit in general

w/ $\displaystyle\int_0^1 X_i(x)\, X_j(x) = 0$ if $i \neq j$

Step 3 • we'll write $\phi(x) = \sum_{n \ge 1} a_n X_n(x)$

• similarly we decompose $f(x,t) = \sum_{n \ge 1} f_n(t)\, X_n(x)$
(eigenfunction expansions)

As always, you compute a_n and $f_n(t)$ by brute force trick. Recall:

$$f(x,t) = \sum_{n\geq 1} f_n(t) X_n(x)$$

$$\int_0^1 f(x,t) X_m(x)\,dx = \sum_{n\geq 1} f_n(t) \int_0^1 X_n(x) X_m(x)\,dx \overset{\text{orthogonality}}{=} f_m(t)\cdot \int_0^1 X_m^2(x)\,dx$$

$$f_m(t) = \frac{\int_0^1 f(x,t) X_m(x)\,dx}{\int_0^1 X_m^2(x)\,dx}$$

<u>Step 4</u>: compute the response to each mode of the source.

We make the usual Ansatz that $u(x,t) = \sum_{n\geq 1} X_n(x) T_n(t)$

where $u_n(x,t) := X_n(x) T_n(t)$ solves the problem

$$\begin{cases} (u_n)_t - \alpha^2 (u_n)_{xx} = f_n(t) X_n(x) \\ \text{usual BCs} \longleftarrow \text{already ok} \\ u_n(x,0) = a_n X_n \end{cases}$$

$$\boxed{u_n(x,t) = X_n(x) T_n(t)}$$

$$\begin{cases} T_n' X_n - \alpha^2 T_n X_n'' = f_n(t) X_n(x) \\ T_n(0) X_n = a_n X_n \end{cases}$$

$$\boxed{X_n'' = -\lambda_n^2 X_n}$$

$$\boxed{\begin{cases} T_n' + \alpha^2 \lambda_n^2 T_n = f_n(t) \\ T_n(0) = a_n \end{cases}} \longleftarrow \text{(Cauchy pb.) for } 1^{st} \text{ order ODE}$$

how to solve? $\begin{cases} y'(t) + a(t) y(t) = b(t) \\ y(t_0) = y_0 \end{cases}$

unique sol. $y(t) = e^{-A(t)}\left[y_0 + \int_{t_0}^{t} e^{A(s)} b(s)\,ds\right]$

$$A(t) = \int_{t_0}^{t} a(s)\,ds$$

So in the special case in question we get

$$T_n(t) = e^{-\alpha^2\lambda_n^2 t}\left[a_n + \int_0^t e^{\alpha^2\lambda_n^2 s} f_n(s)\,ds\right]$$

end of the exercise.

Now, we specify the discussion above to a "concrete case".

$$\begin{cases} u_t - \alpha^2 u_{xx} = \sin(3\pi x) \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = \sin(\pi x) \end{cases}$$

Step 1 (...)

$$\left.\begin{cases} X'' = \mu X \\ X(0) = 0 \\ X(1) = 0 \end{cases}\right\}$$

cf. lecture 3

Dirichlet BC

lead to $\sin(-)$

so standard

Fourier series

already seen $\rightsquigarrow$ $\mu = -\lambda^2$ (step 2)

$$\lambda_n = (n\pi), \quad n \geq 1$$

$$\boxed{X_n(x) = \sin(n\pi x)}$$

Step 3: decompose source and initial datum w.r.t. basis given in step 1

$$f(x,t) = \sin(3\pi x) = \sum_{n\geq 1} \underbrace{f_n(t)}\, X_n(x)$$

$$\rightsquigarrow \quad f_3(t) \equiv 1$$

$$f_n(t) \equiv 0 \quad \text{if } n \neq 3$$

$$\phi(x) = \sin(\pi x) = \sum_{n\geq 1} a_n X_n(x)$$

$$\rightsquigarrow \quad a_1 = 1$$

$$a_n = 0 \quad \text{if } n \neq 1$$

Step 4: response to each mode

$$(\#)\begin{cases} T_n' + (\alpha\pi n)^2 T_n = f_n(t) \\ T_n(0) = a_n \end{cases}$$

① if $n \neq 1, 3$ then (#) becomes

$$\begin{cases} T_n'(t) + (\alpha\pi n)^2 T_n = 0 \\ T_n(0) = 0 \end{cases}$$

$$\longrightarrow \quad T_n \equiv 0$$

② if $n = 1$ then (#) becomes

$$\begin{cases} T_1'(t) + (\alpha\pi)^2 T_1 = 0 \\ T_1(0) = 1 \end{cases}$$

$$\longrightarrow \quad T_1(t) = e^{-(\alpha\pi)^2 t}$$

③ if $n = 3$ then (#) becomes

$$\begin{cases} T_3'(t) + (3\alpha\pi)^2 T_3 = 1 \\ T_3(0) = 0 \end{cases}$$

$$\longrightarrow \quad T_3(t) = e^{-(3\alpha\pi)^2 t} \int_0^t e^{(3\alpha\pi)^2 s}\, ds$$

$$\longrightarrow \quad T_3(t) = \frac{1}{(3\alpha\pi)^2}\left[1 - e^{-(3\alpha\pi)^2 t}\right]$$

Now, we put all pieces together. The solution:

$$u(x,t) = X_1(x)\, T_1(t) + X_3(x)\, T_3(t)$$

$$= e^{-(\alpha\pi)^2 t} \sin(\pi x) + \frac{1}{(3\alpha\pi)^2}\left[1 - e^{-(3\alpha\pi)^2 t}\right] \sin(3\pi x)$$

Asymptotic behaviour for $t \to \infty$

$$u(x,t) \xrightarrow{\sim} \frac{1}{(3\alpha\pi)^2} \sin(3\pi x)$$

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