LECTURE 6

28/10/2021

FOURIER TRANSFORM

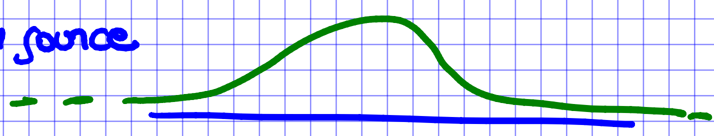
0. Motivation

heat eq. on infinitely long rod

$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R} \\ u(-\infty, t) = 0 \\ u(+\infty, t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Gaussian source

$\phi(x) = e^{-x^2/2}$



Lots of natural applications to WAVE type equations.

1. Heuristics

Recall the complex Fourier series:

$$f \longrightarrow \boxed{FS} \longrightarrow (c_n)_{n \in \mathbb{Z}}$$

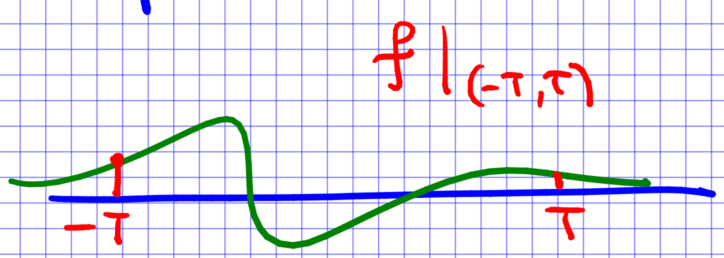
where $$c_n := \frac{1}{2T} \int_{-T}^{T} f(s)\, e^{-\frac{i\pi n s}{T}}\, ds$$

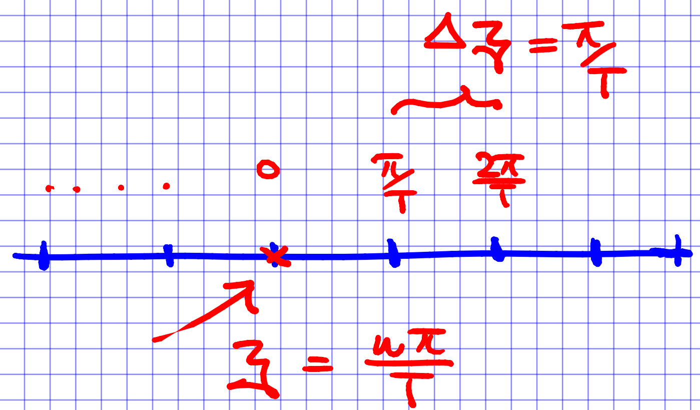
and $$f(x) = \sum_{n \in \mathbb{Z}} c_n\, e^{\frac{i\pi n x}{T}}$$

here $f \in X = \{ \text{piecewise } C^1 \text{ functions } (-T,T) \longrightarrow \mathbb{R} \}$

Q. What happens if we let $T \to \infty$?

We'll obtain the Fourier transform (as a continuous limit of the Fourier series).





$\xi \in \frac{\pi}{T}\mathbb{Z} \xrightarrow[T\to\infty]{\sim} \mathbb{R}$

n-th coefficient of FS of $f|_{(-T,T)}$

If we define $\hat{f}(\xi) := \lim_{T\to\infty} (2T)\, c_n^{(T)} = \int_{-\infty}^{\infty} f(x) e^{-ix\xi}\,dx$

heuristically: why is this a "good" definition?

From $f(x) = \sum_{n\in\mathbb{Z}} c_n^{(T)} e^{i\xi x}$ $\quad$ $T > |x|$

$$= \frac{1}{2\pi} \frac{\pi}{T} \sum_{n\in\mathbb{Z}} 2T c_n^{(T)} e^{i\xi x}$$

this formula is true (for fixed x) for all $T > |x|$, in part.

$$f(x) = \lim_{T\to\infty} \underbrace{\frac{1}{2\pi} \underbrace{\frac{\pi}{T}}_{\Delta\xi} \sum_{n\in\mathbb{Z}} \underbrace{2T c_n^{(T)}}_{\hat{f}(\xi)} e^{i\xi x}}_{\text{Riemann sum for } \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i\xi x}\,d\xi}$$

We expect that if f is nice enough, then

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i\xi x}\,d\xi}$$

2. The key definitions

- Let $f : \mathbb{R} \to \mathbb{R}$ be piecewise C^0 (p.w. continuous) and integrable i.e. $\int_{\mathbb{R}} |f(x)| < \infty$. Then we define its Fourier transform as

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi}\,dx$$

$$f \longrightarrow \boxed{FT} \longrightarrow \hat{f}$$

- Same setting, we define the INVERSE Fourier transform of $g : \mathbb{R} \to \mathbb{R}$ (piecewise C^0, integrable) as

$$\check{g}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} g(\xi) e^{ix\xi} \, d\xi$$

KEY THEOREM: if f is nice (e.g. f is integrable and $\hat{f}$ is also integrable) then

$$\boxed{\mathcal{F}^{-1}[\mathcal{F} f] = f = \mathcal{F}[\mathcal{F}^{-1} f]}$$

(full reconstruction of f from its Fourier transform)

notation:

$$\mathcal{F} f \equiv \hat{f}$$

$$\mathcal{F}^{-1} f \equiv \check{f}$$

---

3. Key properties

(a) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear operators, i.e.

$$\mathcal{F}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{F}[f_1] + c_2 \mathcal{F}[f_2]$$

$$\forall c_1, c_2 \in \mathbb{R} \qquad \forall f_1, f_2$$

(same story for $\mathcal{F}^{-1}$ instead of $\mathcal{F}$)

(b) Derivatives $\longleftrightarrow$ Multiplications

Given f piecewise C^0 and integrable, have

$$(x^n f)^{\wedge} = \left(i \frac{d}{d\xi}\right)^n \hat{f} \qquad (*_1)$$

$$\left(f^{(n)}\right)^\wedge = (i\xi)^n \hat{f} \qquad (*_2)$$

Proof: $(*_1)$ in case $n=1$, else by induction.

$$\left(i\frac{d}{d\xi}\right)\hat{f}(\xi) = \left(i\frac{d}{d\xi}\right)\int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$$

$$= \int_{\mathbb{R}} f(x)\left(i\frac{d}{d\xi}\right)\left(e^{-ix\xi}\right)dx = \int_{\mathbb{R}} i(-ix)\,f(x)e^{-ix\xi}\,dx$$

$$= \int_{\mathbb{R}} \underbrace{x f(x)}\, e^{-ix\xi}\,dx = (x f(x))^\wedge$$

$(*)_2$ in case $n=1$, else by induction.

$$(f')^\wedge = \int_{\mathbb{R}} \underbrace{f'(x)}_{\frac{d}{dx}f(x)} e^{-ix\xi}\,dx$$

dirty way

$$= \Big[\underbrace{f(x)}_{0 \text{ as } x\to\pm\infty}\,\underbrace{e^{-ix\xi}}_{|\ |\leq 1}\Big]_{-\infty}^{+\infty} - \int_{\mathbb{R}} f(x)(-i\xi)e^{-ix\xi}\,dx$$

$$= i\xi\int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$$

to formalise this

$$\int_{-\infty}^{+\infty} f'(x)e^{-ix\xi} = \lim_{n\to\infty}\int_{R_n^-}^{R_n^+} f'(x)e^{-ix\xi}$$

rmk (how to go from dirty to clean)

- If f is C^0 and integrable ($\int|f|<\infty$) it is not true that $\lim_{x\to+\infty} f(x) = 0$

(if it does exist, the limit is 0 but it may not exist)

However: there always exist (R_n^+), (R_n^-) w/ $R_n^+ \to +\infty$, $R_n^- \to -\infty$ s.t.

$$\begin{cases} f(R_n^+) \longrightarrow 0 \\ f(R_n^-) \longrightarrow 0 \end{cases} \text{ as } n\to\infty.$$

Note (corollary of (b)) $\varphi(x) = \int_a^x f(y)\,dy$

$$\rightsquigarrow \quad \hat{\varphi}(\xi) = \frac{1}{i\xi}\hat{f}(\xi)$$

Proof by fund. thm. of calculus $\varphi' = f$

apply FT $\quad \underbrace{(\varphi')^\wedge}_{(i\xi)\hat{\varphi}} = f^\wedge$

$$\Longrightarrow \hat{\varphi} = \frac{1}{i\xi}\hat{f} \qquad \blacksquare$$

(c) TRANSLATIONS $\longleftrightarrow$ MODULATIONS

Let f be piecewise C^0 and integrable, let $a, d \in \mathbb{R}$. ($a \neq 0$)

$$\left(f(a(x-d))\right)^\wedge = \frac{1}{a} e^{-i\xi d} \hat{f}\left(\frac{\xi}{a}\right)$$

Special cases:

$\boxed{a=1}$ pure translation, $\varphi(x) = f(x-d)$

then $\hat{\varphi} = e^{-i\xi d}\hat{f}$

$\boxed{d=0}$ pure dilation, $\varphi(x) = f(ax)$

then $\hat{\varphi}(\xi) = \frac{1}{a}\hat{f}(\xi/a)$

Proof:

Set $\varphi(x) = f(a(x-d))$, then

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x) e^{-ix\xi}\,dx = \int_{\mathbb{R}} f\underbrace{(a(x-d))}_{y} e^{-ix\xi}\,dx$$

Change of variable $x = \frac{y}{a} + d$

$$= \int_{\mathbb{R}} f(y) e^{-i(y/a + d)\xi} \frac{dy}{a} = \frac{1}{a} e^{-i\xi d}\hat{f}(\xi/a) \qquad \blacksquare$$

(d) PRODUCTS

Let f, g be integrable functions $\mathbb{R} \to \mathbb{R}$. Define their convolution product by

$$f * g\,(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy$$

Then

$$(f * g)^\wedge = f^\wedge \cdot g^\wedge \qquad (*_1)$$

and viceversa:

$$(f \cdot g)^\wedge = \frac{1}{2\pi}\, f^\wedge * g^\wedge \qquad (*_2)$$

Proof: $(*_1)$:

$$(f * g)^\wedge (\xi) = \int (f * g)(x)\, e^{-ix\xi}\, dx$$

$$= \int_x \left(\int_y f(y)\, g(x-y)\, dy \right) e^{-ix\xi}\, dx$$

$$= \iint f(y)\, g(x-y)\, e^{-ix\xi}\, dx\, dy$$

Change of variable

$x - y = z$

$$= \iint f(y)\, g(z)\, e^{-i(y+z)\xi}\, dy\, dz$$

$$e^{-i(y+z)\xi} = e^{-iy\xi} \cdot e^{-iz\xi}$$

$$= \left(\int_y f(y)\, e^{-iy\xi}\, dy \right) \left(\int_z g(z)\, e^{-iz\xi}\, dz \right)$$

$$= \hat{f}(\xi) \cdot \hat{g}(\xi) \qquad \blacksquare$$

Proof $(*_2)$: follows from $(*)_1$ ⊕ inversion formula

Computation exactly as above

$$\mathcal{F}^{-1}(f * g) = 2\pi\, f^\vee \cdot g^\vee$$

apply $\mathcal{F}$

$$f * g = 2\pi\, \mathcal{F}(f^\vee \cdot g^\vee) \qquad (\#)$$

if I call $\varphi := f^\vee$, $\psi = g^\vee$ identity (#) above

$$\mathcal{F}(\varphi \cdot \psi) = \frac{1}{2\pi}\left(\varphi^\wedge * \psi^\wedge\right)$$

Example: compute FT of $f(x) = e^{-x^2/2}$

Result: $\hat{f}(\xi) = \sqrt{2\pi}\, f(\xi)$ why?

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-x^2/2} e^{-ix\xi}\, dx$$

$$\frac{d}{d\xi}\hat{f}(\xi) = \int_{\mathbb{R}} \underbrace{(-ix)\, e^{-x^2/2}} e^{-ix\xi}\, dx$$

$$= \int (i) \frac{d}{dx} e^{-x^2/2}\, e^{-ix\xi}\, dx$$

by parts

$\overset{\Downarrow}{=}$ boundary term is zero $- (i) \int_{\mathbb{R}} e^{-x^2/2} (-i\xi) e^{-ix\xi}\, dx$

$$= -\xi \int_{\mathbb{R}} e^{-x^2/2} e^{-ix\xi}\, dx = -\xi\, \hat{f}(\xi)$$

So the function $\hat{f}(\xi)$ satisfies the Cauchy problem

$$\begin{cases} \hat{f}'(\xi) = -\xi\, \hat{f}(\xi) \\ \hat{f}(0) = \sqrt{2\pi} \end{cases} \qquad \frac{d}{d\xi}\log \hat{f}(\xi) = -\xi$$

$$\hat{f}(0) = \int e^{-x^2/2}\, dx$$

$$\Rightarrow \hat{f}(\xi) = \hat{f}(0)\, e^{-\xi^2/2} = \sqrt{2\pi}\, e^{-\xi^2/2}$$

Feynman trick: $J = \int e^{-x^2/2} dx$

$$J^2 = \left(\int e^{-x^2/2} dx\right)\left(\int e^{-y^2/2} dy\right)$$

$$= \iint_{\mathbb{R}^2} e^{-x^2/2} \cdot e^{-y^2/2}\, dx\, dy = \iint_{\mathbb{R}^2} e^{-\frac{(x^2+y^2)}{2}}\, dx\, dy$$

polar

$$= 2\pi \int_0^\infty e^{-\rho^2/2} \rho\, d\rho = 2\pi \left[-e^{-\rho^2/2}\right]\Big|_0^\infty = 2\pi$$