LECTURE 7

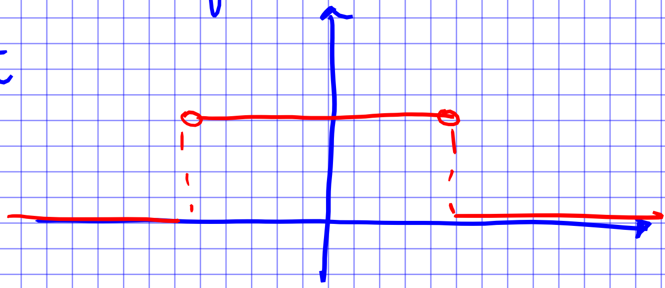
4/11/2021

1st PART: TWO EXERCISES ON FOURIER TRANSFORM

Exercise 1: compute the FT of $f(x) = \begin{cases} 1 & x \in (-T/2, T/2) \\ 0 & \text{else} \end{cases}$

(indicator of the interval $(-T/2, T/2)$).

Plot



Solution: recall the definition $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$

So, in this case,

$$\hat{f}(\xi) = \int_{-T/2}^{T/2} e^{-ix\xi} dx = \left[\frac{1}{(-i\xi)} e^{-ix\xi} \right]_{-T/2}^{T/2} =$$

$$= \frac{1}{(-i\xi)} \left(e^{-iT\xi/2} - e^{iT\xi/2} \right) =$$

$$= \frac{2}{\xi} \left(\frac{e^{iT\xi/2} - e^{-iT\xi/2}}{2i} \right)$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad (\forall z \in \mathbb{C})$$

$$= \frac{2}{\xi} \sin\left(\frac{T\xi}{2}\right)$$

Convenient notation: SINC function defined by

$$\operatorname{sinc}(y) := \frac{\sin(y)}{y} \quad (\hookrightarrow \text{Wikipedia})$$

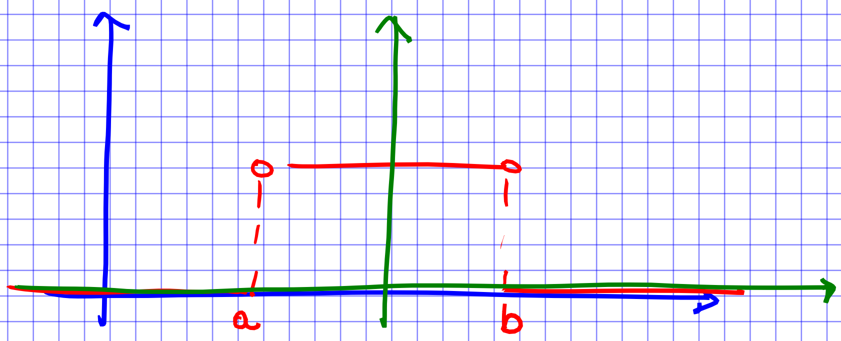
$$\longrightarrow \quad \hat{f}(\xi) = T \operatorname{sinc}\left(\frac{T\xi}{2}\right)$$

bonus question: compute the FT of $\varphi(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & \text{else} \end{cases}$

1st approach: repeat computation above

2nd approach: use the TRANSLATION $\longleftrightarrow$ MODULATION

Plot



Set $C := \underbrace{\frac{a+b}{2}}_{\text{midpoint}}$ $\quad T := \underbrace{b-a}_{\text{length}}$ $\quad \Longrightarrow \varphi(x) = f(x-C)$

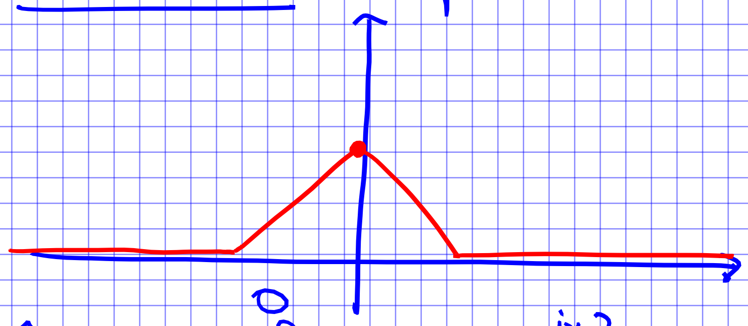
then (by translation $\longleftrightarrow$ modulation) $\quad \hat{\varphi}(\xi) = e^{-iC\xi}\,\hat{f}(\xi)$

Using the explicit form of $\hat{f}$ we get

$$\hat{\varphi}(\xi) = e^{-iC\xi}\, T \operatorname{sinc}\left(\frac{T\xi}{2}\right)$$

$$= e^{-i\left(\frac{a+b}{2}\right)\xi}(b-a)\operatorname{sinc}\left(\frac{(b-a)\xi}{2}\right)$$ ■

Exercise 2: compute the FT of $\quad g(x) = \begin{cases} 1-x & 0<x<1 \\ 1+x & -1<x<0 \\ 0 & \text{else} \end{cases}$



$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x)\, e^{-ix\xi}\,dx$$

$$\hat{g}(\xi) = \int_{-1}^{0} (1+x)\, e^{-ix\xi}\,dx + \int_{0}^{1} (1-x)\, e^{-ix\xi}\,dx$$

$$= \int_{-1}^{1} e^{-ix\xi}\,dx + \int_{-1}^{0} x e^{-ix\xi}\,dx - \int_{0}^{1} x e^{-ix\xi}\,dx$$

$$= \underbrace{2\operatorname{sinc}(\xi)}_{\text{exercise 1, } T=2} + \left[\frac{x}{(-i\xi)}\, e^{-ix\xi}\right]_{-1}^{0} - \int_{-1}^{0} \frac{e^{-ix\xi}}{(-i\xi)}\,dx$$

$$- \left[\frac{x}{(-i\xi)}\, e^{-ix\xi}\right]_{0}^{1} + \int_{0}^{1} \frac{e^{-ix\xi}}{(-i\xi)}\,dx$$

$$= \cancel{2\,\mathrm{sinc}(\xi)} + \frac{1}{(-i\xi)}e^{i\xi} + \frac{1}{(i\xi)}\left[\frac{e^{-ix\xi}}{(-i\xi)}\right]_{-1}^{0}$$

$$- \frac{1}{(-i\xi)}e^{-i\xi} \underbrace{- \frac{1}{(i\xi)}\left[\frac{e^{-ix\xi}}{(-i\xi)}\right]_{0}^{1}}$$

$$= -\cancel{2\,\mathrm{sinc}(\xi)}$$

$$= \frac{1}{\xi^2}\left[1 - e^{i\xi} - e^{-i\xi} + 1\right]$$

<u>Conclusion</u>: $$\hat{g}(\xi) = \frac{1}{\xi^2}\left(2 - e^{i\xi} - e^{-i\xi}\right)$$

Note $$\mathrm{sinc}^2(\xi/2) = \left\{\frac{\sin(\xi/2)}{(\xi/2)}\right\}^2 =$$ use $\sin(\cdot)$ in terms of complex exp

$$= \frac{4}{\xi^2}\left(\frac{e^{i\xi/2} - e^{-i\xi/2}}{2i}\right)^2$$

$$= -\frac{1}{\xi^2}\left(e^{i\xi} + e^{-i\xi} - 2\right) = \frac{1}{\xi^2}\left(2 - e^{i\xi} - e^{-i\xi}\right)$$

$$\Longrightarrow \quad \hat{g}(\xi) = (\hat{f}(\xi))^2$$

can you solve the mistery?

<u>Solution</u> of <u>mistery</u>: recall the <u>convolution</u> operation

$$f_1 * f_2(x) = \int_{\mathbb{R}} f_1(y)\, f_2(x-y)\,dy$$

Getting to exercise 2, one checks (problem 7.1)

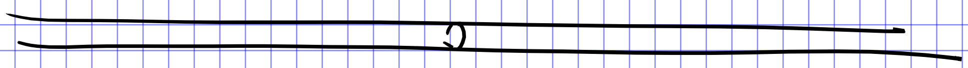
$$\boxed{f * f = g}$$

$$\text{FT} \hookrightarrow \quad (f*f)^\wedge = g^\wedge \quad \longrightarrow \quad \hat{f}\cdot\hat{f} = \hat{g}$$

$$\Longleftrightarrow \quad (\hat{f})^2 = \hat{g}$$

2nd part: using the FT to solve PDEs

toy model: heat eq. on an infinitely long rod



$x \in \mathbb{R}$

$t \geq 0$

$u(x,t)$

$$\begin{cases} u_t = c^2 u_{xx} & (*) \\ u(x,0) = f(x) \end{cases}$$

For instance $f(x) = \begin{cases} T & 0 < x \; 1 \\ -T & -1 < x < 0 \end{cases}$

Key trick: apply FT to PDE $(*)$ w.r.t. variable x,

i.e. set $\qquad \hat{u}(\xi, t) := \int_{\mathbb{R}} u(x,t)\, e^{-ix\xi}\, dx$

then $\qquad (u_t)^\wedge = c^2 \, (u_{xx})^\wedge$

$$(u_t)^\wedge = \int_{\mathbb{R}} u_t(x,t)\, e^{-ix\xi}\, dx = \frac{d}{dt} \int_{\mathbb{R}} u(x,t)\, e^{-ix\xi}\, dx$$

$$= \frac{d}{dt} \hat{u}$$

$$(u_{xx})^\wedge = (i\xi)^2 \, \hat{u}$$

DERIVATION $\longleftrightarrow$ MULTIPLICATION

So, after transforming, our PDE $(*)$ becomes the ODE

$$\frac{d}{dt} \hat{u} = c^2 (i\xi)^2 \hat{u} = -c^2 \xi^2 \hat{u}$$

Similarly, one transforms the initial condition into

$$\hat{u}(\xi, 0) = \hat{f}(\xi)$$

Put things together

$$\begin{cases} \frac{d}{dt} \hat{u} = -c^2 \xi^2 \hat{u} \\ \hat{u}(\xi, 0) = \hat{f}(\xi) \end{cases}$$

This can be integrated explicitly, and we get

$$\boxed{\hat{u}(\xi, t) = \hat{f}(\xi)\, e^{-c^2\xi^2 t}}$$

Abstractly speaking, this solves our problem since (seen last time) "a function is uniquely determined by its Fourier transform"

$$u(x,t) = \frac{1}{2\pi}\int \hat{u}(\xi,t)\, e^{ix\xi}\, d\xi$$

$$= \mathcal{F}^{-1}\left[\hat{u}(\xi,t)\right]$$

$$= \mathcal{F}^{-1}\left[\hat{f}(\xi)\cdot \kappa(\xi)\right]$$

Set $\kappa(\xi) = e^{-c^2\xi^2 t}$

$$= \mathcal{F}^{-1}\left[\hat{f}(\xi)\right] * \mathcal{F}^{-1}\left[\kappa(\xi)\right]$$

$$= f(x) * \mathcal{F}^{-1}\left[\kappa(\xi)\right]$$

To compute $\mathcal{F}^{-1}\left[\kappa(\xi)\right]$ need to recall computation of FT of a gaussian. Specifically:

$$\mathcal{F}^{-1}[\kappa] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-c^2\xi^2 t}\, e^{ix\xi}\, d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(2c^2t)\xi^2/2}\cdot e^{ix\xi}\, d\xi$$

$$u^2 = (2c^2t)\,\xi^2 \qquad = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-u^2/2}\, e^{ix\frac{u}{\sqrt{2c^2t}}}\, \frac{du}{\sqrt{2c^2t}}$$

$$= \frac{1}{2\pi\sqrt{2c^2t}} \times \left\{ \begin{array}{l} \text{Fourier transform of } u \longmapsto e^{-u^2/2} \\ \text{evaluated at } -\frac{x}{\sqrt{2c^2t}} \end{array} \right\}$$

By computation seen at end of Lecture 6

$$= \frac{1}{2\pi\sqrt{2c^2t}} \sqrt{2\pi}\, e^{-\frac{x^2}{2}\frac{1}{2c^2t}}$$

$$= \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-\frac{x^2}{4c^2t}} =: K(x,t)$$

heat kernel

Conclusion: we have written the solution to

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \end{cases}$$

in the form $u(x,t) = f(x) * K(x,t)$

so, in explicit terms,

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4c^2t}} f(y)\, dy}$$