LECTURE 8 11/11/2021

1st part: some facts about heat propagation on the real line

$$\begin{cases} u_t = c^2 u_{xx} & t \geq 0 \,,\ x \in \mathbb{R} \\ u(x,0) = f(x) \end{cases}$$

known function

Using FT we wrote the solution to this problem

$u(x,t) = \{$ convolution of the datum f with the heat kernel $\}$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4c^2t}} f(y)\,dy$$

Q. What physical information can we extract from here?

(1) FAST DIFFUSION PROPERTY:

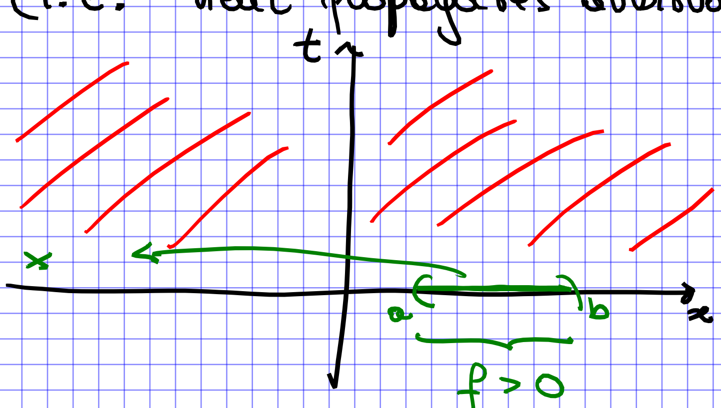
Suppose $f \geq 0$, f vanishes outside some interval $I = (a,b)$

i.e. $f(x) > 0$ for $x \in (a,b)$

$f(x) = 0$ for $x \notin (a,b)$

Then $u(x,t) > 0$ for all $t > 0$ and all $x \in \mathbb{R}$.

(i.e. "heat propagates arbitrarily fast")



Rmk: this conclusion is in strong contrast to wave equation, where FINITE SPEED propagation occurs.

Reason for (1):

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{\mathbb{R}} \underbrace{e^{-\frac{(x-y)^2}{4c^2t}} f(y)}_{\substack{\geq 0 \quad \forall y \in \mathbb{R} \\ > 0 \quad \text{if } y \in (a,b)}}\,dy$$

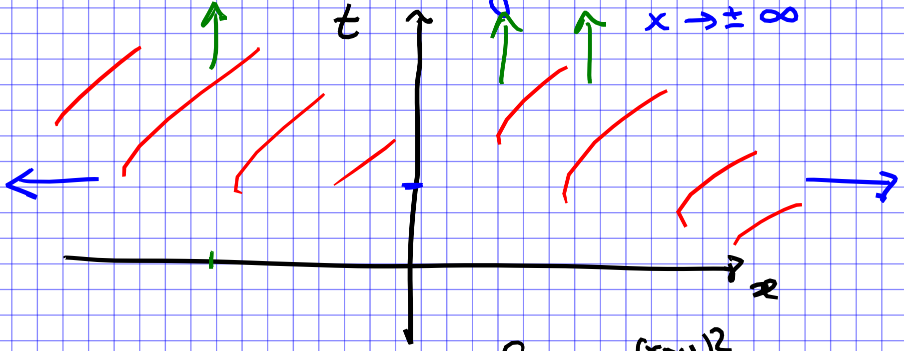
Analysis I $\longrightarrow$ $u(x,t) > 0$

② ASYMPTOTIC DECAY

Let's study a special case $f(x) = \begin{cases} T & -1 < x < 1 \\ 0 & \text{else} \end{cases}$

- Q: what can we say about $\lim_{t \to +\infty} u(x,t)$ for any fixed x?

- Q: What can we say about $\lim_{x \to \pm\infty} u(x,t)$ for any fixed t?



$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4c^2t}} f(y)\,dy$$

$$= \frac{T}{\sqrt{4\pi c^2 t}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4c^2t}}\,dy$$

$$= \frac{T}{\sqrt{4\pi c^2 t}} \int_{\frac{x-1}{\sqrt{2c^2t}}}^{\frac{x+1}{\sqrt{2c^2t}}} e^{-u^2/2} \sqrt{2c^2t}\,du$$

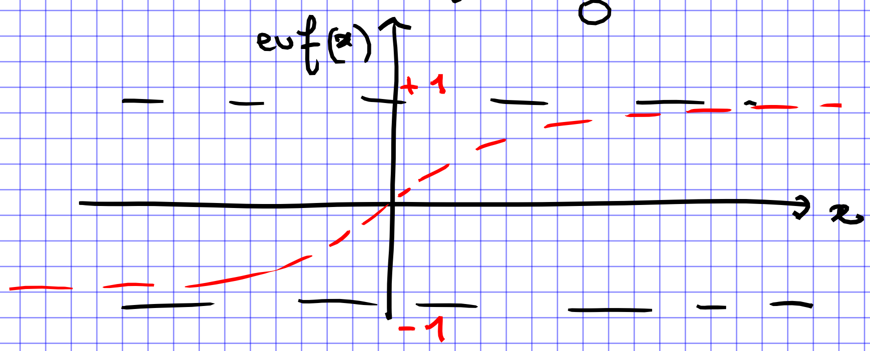
$y \longmapsto u$

$u = \dfrac{x-y}{\sqrt{2c^2t}}$

$$= \frac{T}{\sqrt{2\pi}} \int_{\frac{x-1}{\sqrt{2c^2t}}}^{\frac{x+1}{\sqrt{2c^2t}}} e^{-u^2/2}\,du$$

To proceed, let's introduce the GAUSS ERROR FUNCTION

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2}\,dv$$



$$u(x,t) = T\left\{ \underbrace{\frac{1}{\sqrt{2\pi}} \int_0^{\frac{x+1}{\sqrt{2c^2t}}} e^{-u^2/2}\,du}_{(I)} - \underbrace{\frac{1}{\sqrt{2\pi}} \int_0^{\frac{x-1}{\sqrt{2c^2t}}} e^{-u^2/2}\,du}_{(II)} \right\}$$

change of variable

$$\frac{u}{\sqrt{2}} = v$$

$$(I) = \frac{1}{\sqrt{2\pi}} \int_0^{\frac{x+1}{2\sqrt{c^2t}}} e^{-v^2}\sqrt{2}\,dv = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x+1}{2\sqrt{c^2t}}} e^{-v^2}\,dv$$

$$= \frac{1}{2}\,\mathrm{erf}\left(\frac{x+1}{2\sqrt{c^2t}}\right)$$

$$(II) = \frac{1}{2}\,\mathrm{erf}\left(\frac{x-1}{2\sqrt{c^2t}}\right)$$

$$u(x,t) = T\left\{ \frac{\mathrm{erf}\left(\frac{x+1}{2\sqrt{c^2t}}\right) - \mathrm{erf}\left(\frac{x-1}{2\sqrt{c^2t}}\right)}{2} \right\}$$

Let's answer the two questions above:

1st q. Fix x and study asymptotic behaviour in t.

For any fixed x, have $\lim_{t \to +\infty} \mathrm{erf}\left(\frac{x+1}{2\sqrt{c^2t}}\right) = 0$

$$\lim_{t \to +\infty} \mathrm{erf}\left(\frac{x-1}{2\sqrt{c^2t}}\right) = 0$$

2nd q. Fix $t > 0$ and study asymptotic behaviour in x

For any fixed t, have $\lim_{x \to +\infty} \mathrm{erf}\left(\frac{x+1}{2\sqrt{c^2t}}\right) = 1$

$$\lim_{x \to +\infty} \mathrm{erf}\left(\frac{x-1}{2\sqrt{c^2t}}\right) = 1$$

$\lim_{x \to +\infty} u(x,t) = 0$ same story $\lim_{x \to -\infty} u(x,t) = 0$

Two additional properties:

③ L^1- conservation law

If f is integrable, i.e. $\int_{\mathbb{R}} |f(x)|\,dx < \infty$

then the quantity $\bar{u}(t) := \int_{\mathbb{R}} u(x,t)\,dx$

is CONSTANT in time $\left(= \int_{\mathbb{R}} f(x)\,dx \right)$

~ ∘ ~

④ L^2- dissipation law

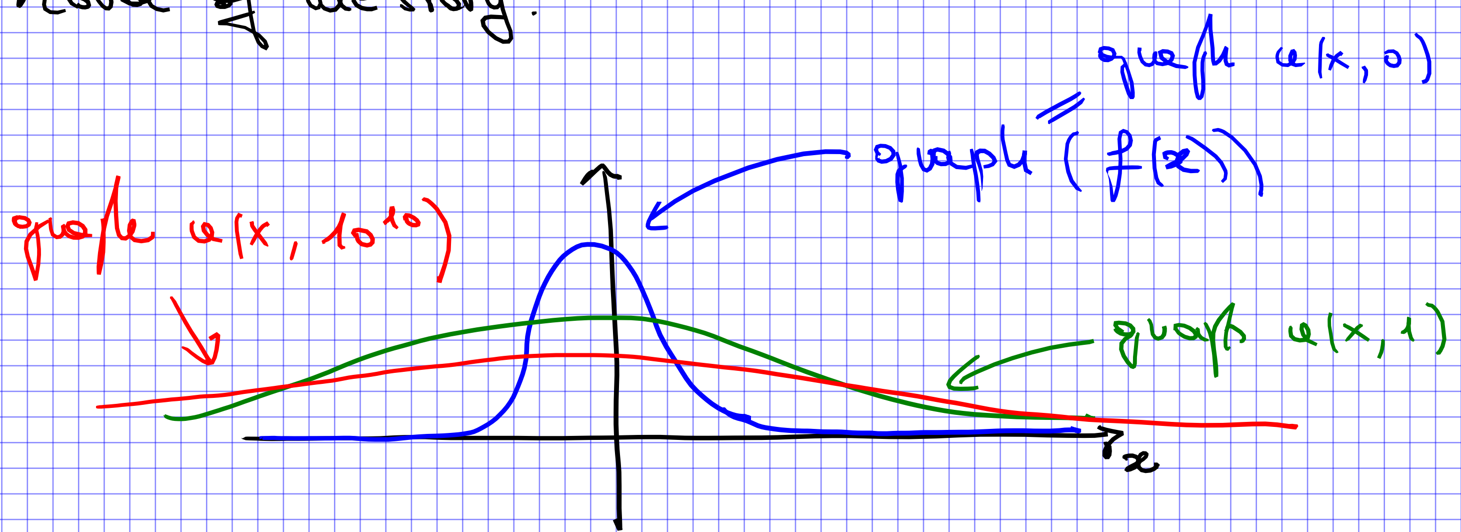
If f is integrable i.e. $\int_{\mathbb{R}} |f(x)|\,dx < \infty$
and square-integrable $\int_{\mathbb{R}} |f^2(x)|\,dx < \infty$

then the quantity $\bar{u}_2(t) := \int_{\mathbb{R}} u^2(x,t)\,dx$

is MONOTONE ↘ in time $\left(\leq \bar{u}_2(0) = \int_{\mathbb{R}} f^2(x)\,dx \right)$

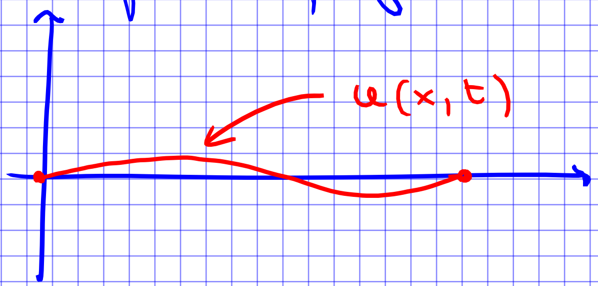
~ ∘ ~

Moral of the story:

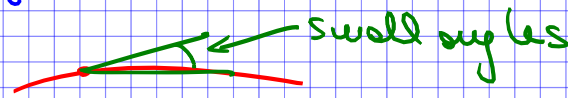


2nd part: (ONE-D) WAVE EQUATION

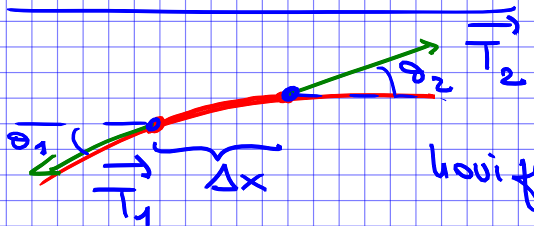
Simplest physical model: violin string



hp.
- thin string
- homogeneous material
- planar problem
- "small oscillations"



Newton's 2nd law: ← ignore effects of gravity



horizontal balance $\quad T_2 \cos\theta_2 = T_1 \cos\theta_1$

vertical balance $\quad (\rho \Delta x)\, u_{tt} = T_2 \sin\theta_2 - T_1 \sin\theta_1$

$$\cos\alpha = 1 + O(\alpha^2)$$

$\hookrightarrow$ horizontal balance $\quad T_2 = T_1 + O(\Delta x^2)$

$$\hookrightarrow \sin\theta_2 - \sin\theta_1 = \sin(\theta(x+\Delta x)) - \sin(\theta(x))$$

$$\overset{\text{small angle}}{=} \tan(\theta(x+\Delta x)) - \tan(\theta(x)) + O(\Delta x^2)$$

$$= u_x(x+\Delta x) - u_x(x) + O(\Delta x^2)$$

$$\overset{\text{Taylor to } u_x}{=} \cancel{u_x(x)} + u_{xx}(x)\Delta x + O(\Delta x^2) - \cancel{u_x(x)} - O(\Delta x^2)$$

$$= u_{xx}(x)\,\Delta x + O(\Delta x^2)$$ ← plug-in these info in vertical balance

Rmk. $O(y^2)$ means "collection of all terms in Taylor's formula of order at least 2 in y"

$$(\lambda \Delta x)\, u_{tt} = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

$$= (\underbrace{T_1}_{T} + O(\Delta x^2)) \sin\theta_2 - \underbrace{T_1}_{} \sin\theta_1$$

$$= T\,(\sin\theta_2 - \sin\theta_1) + O(\Delta x^2)$$

$$= T\,(u_{xx}(x)\,\Delta x + O(\Delta x^2)) + O(\Delta x^2)$$

$$= T\, u_{xx}(x)\, \Delta x + O(\Delta x^2)$$

$$u_{tt} = c^2 u_{xx}(x) + \underbrace{O(\Delta x)}_{\longrightarrow 0 \text{ as } \Delta x \to 0}$$

$$\Longrightarrow \boxed{u_{tt} = c^2 u_{xx}}$$ 1 DIM'L WAVE EQUATION

~ ~

IBVP for wave equation

- String of length L, same physical hp. as above
- general IBVP

$$\begin{cases} u_{tt} = c^2 u_{xx} + \boxed{} \\ \left.\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array}\right\} \text{initial conditions} \\ \left.\begin{array}{l} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = b_1(t) \\ \alpha_2 u_x(L,t) + \beta_2 u(L,t) = b_2(t) \end{array}\right\} \text{boundary conditions} \end{cases}$$

- Special cases for homogeneous BC

Dirichlet $\quad \alpha_1 = \alpha_2 = 0 \quad \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$

(typical violin string)

Neumann $\quad \beta_1 = \beta_2 = 0 \quad \begin{cases} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{cases}$

(for physical interpretation of Neumann BC for wave eq. see Lesson 19 in Farlow)

- **General methodology** : separation of variables

$$u(x,t) = \sum_{n \geq 0} X_n(x)\, T_n(t)$$

where X_n, T_n are gotten by solving ODEs

examples (to be discussed in class) : problem 8.1 and 8.2