Lecture 9 18/11/2021

1D WAVE EQUATION ON INFINITE STRING

$\longrightarrow$ D'Alembert formula

Problem: solve this initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < +\infty \\ u(x,0) = f(x) & u(x,t) \\ u_t(x,0) = g(x) \end{cases}$$

General methodology: apply the Fourier transform in the variable x

$$\left.\begin{aligned} \hat{u}(\xi,t) &= \int_{\mathbb{R}} u(x,t)\, e^{-i\xi x}\, dx \\ (u_{xx})^\wedge &= (i\xi)^2 \hat{u} = -\xi^2 \hat{u} \\ (u_{tt})^\wedge &= \frac{d^2}{dt^2}\hat{u} \end{aligned}\right\} \text{rewrite the problem}$$

$$\begin{cases} (\hat{u})'' = -c^2\xi^2 \hat{u} \\ \hat{u}(\xi,0) = \hat{f}(\xi) \\ (\hat{u})'(\xi,0) = \hat{g}(\xi) \end{cases} \qquad \text{here: } (\hat{u})' \equiv \frac{d}{dt}\hat{u}$$

Solution of the sole ODE $(\hat{u})'' = -c^2\xi^2\hat{u}$

$$\hat{u}(\xi,t) = A(\xi)\cos(c\xi t) + B(\xi)\sin(c\xi t)$$

now $A(\xi), B(\xi)$ are determined using initial conditions

$$\begin{cases} \hat{u}(\xi,0) = A(\xi) = \hat{f}(\xi) \\ (\hat{u})'(\xi,0) = c\xi B(\xi) = \hat{g}(\xi) \end{cases}$$

$$\rightsquigarrow \begin{cases} A(\xi) = \hat{f}(\xi) \\ B(\xi) = \dfrac{\hat{g}(\xi)}{c\,\xi} \end{cases}$$

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(\xi,t)\, e^{i\xi x}\, d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\hat{f}(\xi)\cos(c\xi t) + \hat{g}(\xi)\frac{\sin(c\xi t)}{c\xi} \right) e^{i\xi x}\, d\xi$$

$$= \underbrace{\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\cos(c\xi t)\, e^{i\xi x}\, d\xi}_{(\mathrm{I})} + \underbrace{\frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\frac{\sin(c\xi t)}{c\xi}\, e^{i\xi x}\, d\xi}_{(\mathrm{II})}$$

<u>Treatment of (I)</u>:

$$(\mathrm{I}) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\cos(c\xi t)\, e^{i\xi x}\, d\xi \qquad \cos(p) = \frac{e^{ip}+e^{-ip}}{2}$$

$p \rightsquigarrow c\xi t$

$$= \frac{1}{2}\left\{ \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\, e^{ic\xi t}\cdot e^{i\xi x}\, d\xi + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\, e^{-ic\xi t}\, e^{i\xi x}\, d\xi \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\, e^{i(x+ct)\xi}\, d\xi + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\, e^{i(x-ct)\xi}\, d\xi \right\}$$

$$= \frac{1}{2}\left\{ f(x+ct) + f(x-ct) \right\}$$

<u>Treatment of (II)</u>:

$$(\mathrm{II}) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\frac{\sin(c\xi t)}{c\xi}\, e^{i\xi x}\, d\xi$$

$$\left[\text{TRICK} \quad \frac{\sin(c\xi t)}{c\xi} = \int_0^t \cos(c\xi s)\, ds \right]$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\left(\int_0^t \cos(c\xi s)\, ds \right) e^{i\xi x}\, d\xi$$

$$= \int_0^t \Bigg(\underbrace{\frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\cos(c\xi s)\, e^{i\xi x}\, d\xi}_{= \frac{1}{2}\left(g(x+cs)+g(x-cs)\right)} \Bigg)\, ds$$

interchange order of integration

as computed for (I)

$$= \frac{1}{2}\int_0^t \left(g(x+cs) + g(x-cs) \right) ds$$

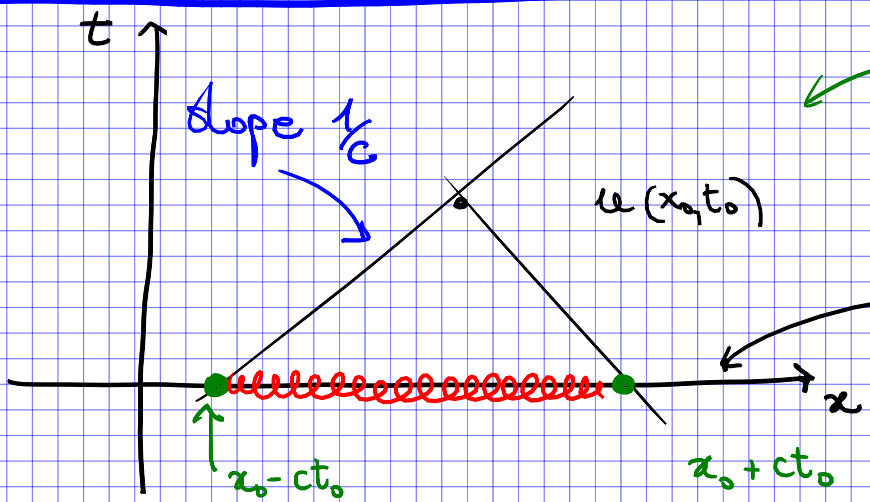
$$= \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$$

Analysis 1
change of variable

Putting together (I) and (II) we have proven D'Alembert formula : the solution to our problem is

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du}$$

picture: assume $c=1$



this is the line where we assign the initial data
$u(x,0) = f(x)$
$u_t(x,0) = g(x)$

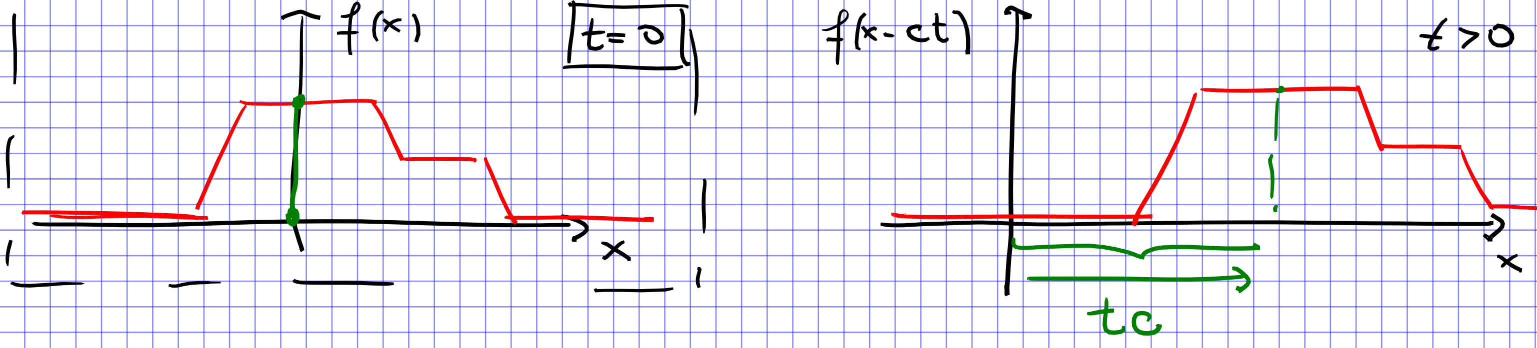
Finite speed of propagation $= c$ (compare w/ heat equation)

Evolution of fixed-time profile of the solution

Assume $g = 0$ so
$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct) \right)$$

- physical meaning of $f(x-ct)$

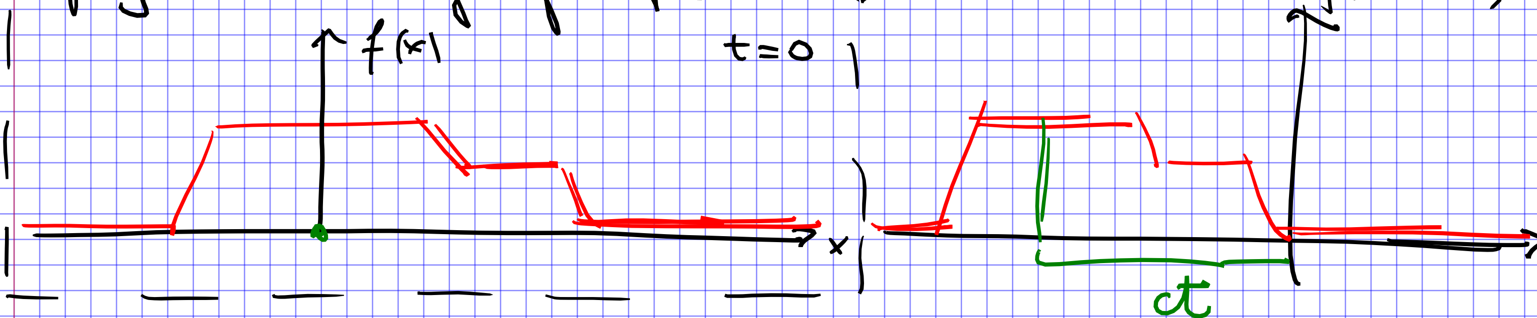


as t increases, the graph of the function

$$x \longmapsto f(x-ct)$$

moves to the RIGHT w/ speed c.

- physical meaning of $f(x+ct)$



as t increases, the graph of the function

$$x \longmapsto f(x+ct)$$

moves to the LEFT w/ speed c.

<u>Exercise</u>

① Let $f(x) = \begin{cases} 3 & -1 < x < 2 \\ 0 & \text{else} \end{cases}$

Let u solve $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$ Plot $u(x,1)$ and $u(x,100)$.

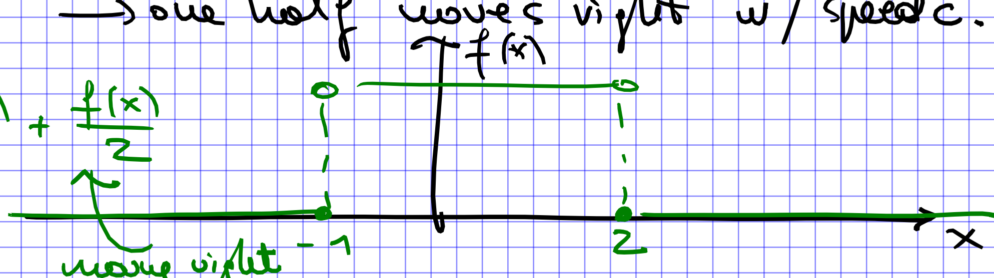
<u>Recipe</u>: • divide the initial displacement (f) by half

→ one half moves left w/ speed c

→ one half moves right w/ speed c.

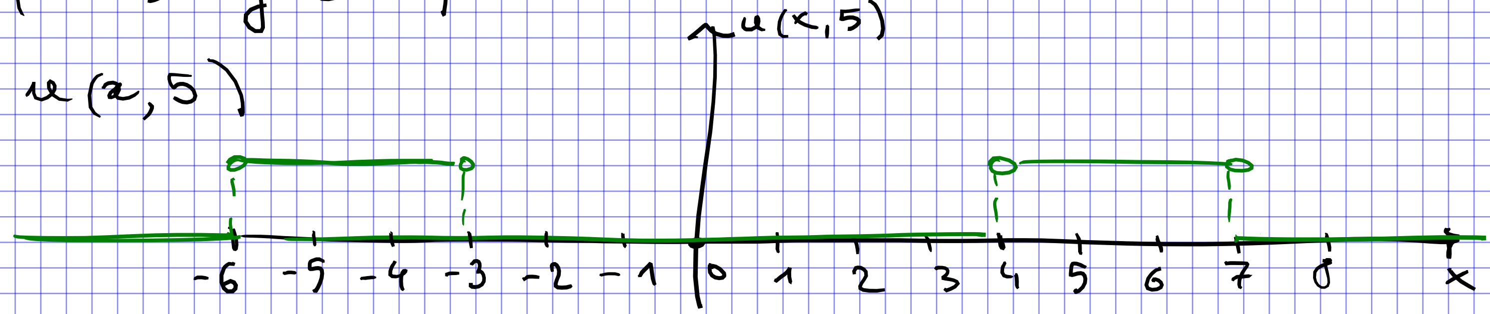
$f(x) = \frac{f(x)}{2} + \frac{f(x)}{2}$

moving left   moving right



(let's say $c=1$)

$u(x,5)$



- Reality check for you: plot $u(x,2)$.
- Reality check: Same questions but w/ $f(x) = e^{-x^2}$.

Canonical coordinates for wave equations

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \xrightarrow[\text{trick}]{} \begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases} \quad (*)$$

(no relation to Fourier transform)

Set $w(\xi, \eta) = u(x(\xi,\eta), y(\xi,\eta))$

$$= u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$$

why? $\quad (*) \iff \begin{cases} x = \dfrac{\xi+\eta}{2} \\ y = \dfrac{\xi-\eta}{2} \end{cases}$

Get a PDE problem for function w. (Kettenregel)

For instance: $\quad u_t = \dfrac{\partial u}{\partial t} = \dfrac{\partial w}{\partial \xi} \underbrace{\dfrac{\partial \xi}{\partial t}}_{c} + \dfrac{\partial w}{\partial \eta} \underbrace{\dfrac{\partial \eta}{\partial t}}_{-c}$

$$= c\,(w_\xi - w_\eta)$$

Similar procedure:

$$\begin{cases} u_{tt} = c^2 (w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}) \\ u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \end{cases}$$

$$u_{tt} = c^2 u_{xx} \iff -4c^2 w_{\xi\eta} = 0$$

$$\iff \boxed{w_{\xi\eta} = 0}$$

<u>Fact</u>: $w(\xi, \eta) = A(\xi) + B(\eta)$

↳ now transform back in u, this equation is saying

$$u(x,t) = A(x+ct) + B(x-ct)$$

how do you determine A, B? By using IC

$$\begin{cases} u(x,0) = A(x) + B(x) = f(x) \\ u_t(x,0) = cA'(x) - cB'(x) = g(x) \end{cases}$$

Solving for A, B. Integrate 2nd equation:

$$\begin{cases} A(x) - B(x) = \dfrac{1}{c}\displaystyle\int_{x_0}^{x} g(u)\,du + k \\ A(x) + B(x) = f(x) \end{cases}$$

$$\begin{cases} A(x) = \dfrac{1}{2}\left(f(x) + \dfrac{1}{c}\displaystyle\int_{x_0}^{x} g(u)\,du + k \right) \\ B(x) = \dfrac{1}{2}\left(f(x) - \dfrac{1}{c}\displaystyle\int_{x_0}^{x} g(u)\,du - k \right) \end{cases}$$

Lastly:

$$u(x,t) = A(x+ct) + B(x-ct)$$

$$= \frac{1}{2}\left(f(x+ct) + \frac{1}{c}\int_{x_0}^{x+ct} g(u)\,du \cancel{+k} \right)$$

$$+ \frac{1}{2}\left(f(x-ct) - \frac{1}{c}\int_{x_0}^{x-ct} g(u)\,du \cancel{-k} \right)$$

$$= \frac{1}{2}\big(f(x+ct) + f(x-ct) \big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$$

∎