Configurations on Elliptic Curves

Dedicated to the memory of Branko Grünbaum

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Abstract

An elliptic configuration is a configuration with all its points on a cubic curve. We investigate the existence of elliptic $(3r_4, 4r_3)$-configurations for $r \geq 5$ and show that for every $k \geq 2$ there is an elliptic $(9k_4, 12k_3)$-configuration with a rotational symmetry of order 3.

1 Terminology

A (p_λ, l_π)-*configuration* consists of p points and l lines in the real affine plane such that each point belongs to λ lines and each line goes through π points. If $p = l$ and consequently $\lambda = \pi$, we just write (p_λ) instead of (p_λ, l_π). A configuration is called an *elliptic configuration* if there is a cubic curve which passes through all points of the configuration (see also the discussion of elliptic configurations in Grünbaum [2, p. 247 ff.]). An example of an elliptic $(12_4, 16_3)$-configuration is given in Grünbaum [2, p. 249].

For a finite group G, a configuration is called G-*symmetric* if G is a subgroup of the symmetry group of the configuration. Finally, an *elliptic G-symmetric configuration* is a configuration which is both, elliptic and G-symmetric.

Since a line intersects a cubic curve in at most 3 different points, the maximum value for π of an elliptic (p_λ, l_π)-configuration is $\pi = 3$, and therefore, natural candidates for elliptic configurations are $(3r_3)$-configurations and $(3r_4, 4r_3)$-configurations for $r \geq 1$ (for $(12_4, 16_3)$-configurations see, for example, Gropp [1] or Metelka [5]). On page 293 of Grünbaum [2], *Open Problem 4* asks to decide for which $r \geq 5$ elliptic $(3r_4, 4r_3)$-configurations exist.

Of particular interest are elliptic configurations with C_3 or D_3 symmetry. Here, D_3 is the dihedral group of the regular triangle, and C_3 its subgroup of of elements of odd order. For $G = D_3$ or $G = C_3$ the number of lines of a G-symmetric configuration must be a multiple of 3. Hence, since $3 \mid 4r$ implies $3 \mid r$, the possible elliptic D_3 or C_3-symmetric $(3r_4, 4r_3)$-configurations are $(9k_4, 12k_3)$-configurations for $k \geq 1$.

After introducing a normal form of cubic curves which are D_3-symmetric, we give a construction of elliptic D_3-symmetric $(9k_4, 12k_3)$-configurations for every $k \geq 2$. Finally, we show the existence of elliptic $(3r_4, 4r_3)$-configurations for some $r \geq 5$. The constructions of elliptic configurations are motivated by Schroeter's ruler construction of cubic curves (see [4] for some first examples of elliptic configurations).

2 A D_3-symmetric normal form for cubic curves

It is well-known that every non-singular cubic curve in the real projective plane can be transformed into Weierstrass Normal Form

$$y^2 = x^3 + ax^2 + bx.$$

Without loss of generality, we may require that the x-coordinate of an inflection point is 1. In this case we get (see [3, Fact 2.3])

$$b \neq 1 \quad \text{and} \quad a = \frac{b^2 - 6b - 3}{4}. \tag{1}$$

Now, by computing the polar conic at the point $(0, 1, 0)$ in the projective extension of the plane as well as the intersection points of the tangents at the inflections points, we find the projective transformation

$$\begin{pmatrix} 1 & 0 & -2b \\ 0 & \frac{\sqrt{3}(b-1)}{2} & 0 \\ 1 & 0 & b-3 \end{pmatrix}$$

which transforms the affine curve $y^2 = x^3 + ax^2 + bx$ (with a, b as in (1)) into the curve

$$\Gamma_{D_3}: \ x^3 - 3xy^2 - 3(b-3)(x^2 + y^2) + 4b^2(b-9) \ = \ 0.$$

To see that the latter curve is D_3-symmetric, notice first that the curve is symmetric with respect to the x-axis. To see that the curve is also symmetric with respect to rotations about the origin with angle $\frac{2\pi}{3}$, notice that if (x_0, y_0) is a point on the curve Γ_{D_3}, then also

$$\begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ -\sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is a point on Γ_{D_3}. Figure 1 shows two D_3-symmetric curves Γ_{D_3}.

Conic sections have a natural reflection symmetry along their axes. It is quite natural to look at cubic curves in a D_3-symmetric form. In this regard, we now have:

Proposition 1. *Every regular cubic curve can be brought into the D_3-symmetric normal form*

$$\Gamma_{D_3}: \; x^3 - 3xy^2 - 3(b-3)(x^2+y^2) + 4b^2(b-9) \;=\; 0$$

with $b \in \mathbb{R} \setminus \{1\}$.

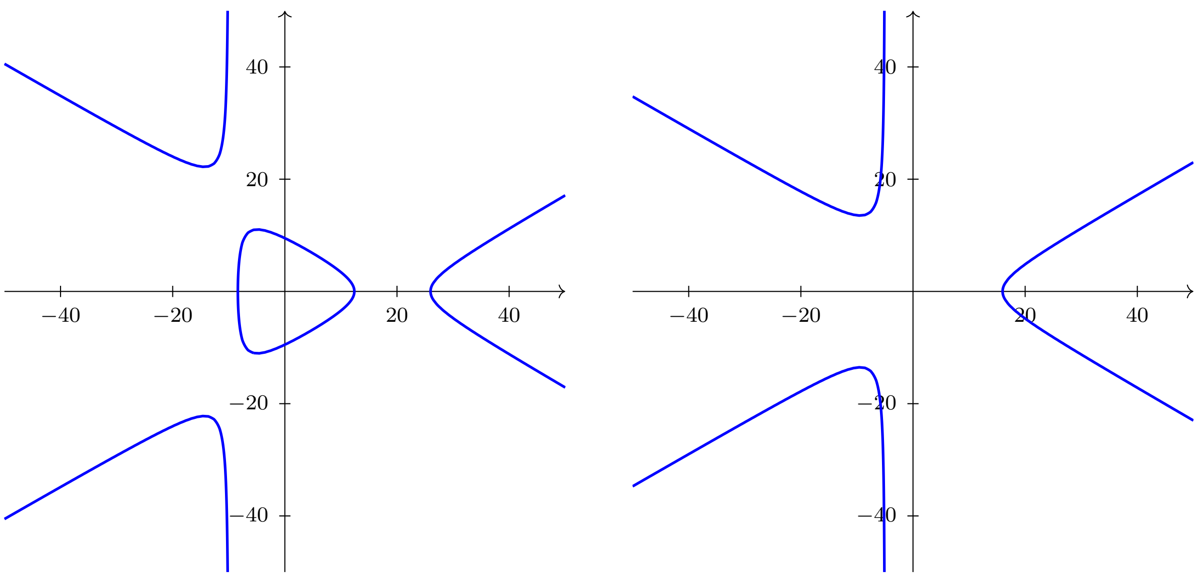


Figure 1: Elliptic D_3-symmetric curves for $b = 13$ (left), and $b = 8$ (right).

3 Elliptic D_3-symmetric $(9k_4, 12k_3)$-configurations

In order to construct an elliptic D_3-symmetric $(9k_4, 12k_3)$-configuration for some $k \geq 2$, we take an arbitrary D_3-symmetric elliptic curve Γ_0 with neutral element $\mathscr{O} = (0, 1, 0)$ and choose a point Q on Γ_0 of order $9k + 3$. This can be achieved by considering a p-periodic parametrization of the curve by the Weierstrass $\wp$-function and taking the point Q as the image of the parameter value $\frac{pq}{9k+3}$ for some q with $(q, 9k+3) = 1$. The group G_k on Γ_0, generated by the point Q, is isomorphic to the group $\mathbb{Z}/(9k+3)\mathbb{Z}$. For $1 \leq i \leq 9k+3$, let

$$P_i := i * Q := \underbrace{Q + Q + \ldots + Q}_{i \text{ terms}},$$

where we denote the group operation on Γ_0 by $+$. We define the following three sets of points:

$$S_0 := \{P_1, \ldots, P_{3k}\}, \qquad S_1 := \{P_{3k+2}, \ldots, P_{6k+1}\}, \qquad S_2 := \{P_{6k+3}, \ldots, P_{9k+2}\}$$

Then each S_j (for $j \in \{0,1,2\}$) contains $3k$ pairwise distinct points, and since the sets S_j are pairwise disjoint, the set $S := S_0 \cup S_1 \cup S_2$ contains $9k$ pairwise distinct points on the curve Γ_0. Notice that since the points P_{3k+1}, P_{6k+2} of order three, and P_{9k+3} are the only points of Γ_0 at infinity and neither of them belongs to S, all points of S belong to the real affine plane. The goal is now to construct a D_3-symmetric, $(9k_4, 12k_3)$-configuration on the set of points S. Before we start with the construction, let us introduce some notation.

- We identify the group G_k with the group $\mathbb{Z}/(9k+3)\mathbb{Z}$, and for $1 \leq u \leq 9k+3$, we identify the point P_u with $u \in G_k$ (*i.e.*, with an element in $\mathbb{Z}/(9k+3)\mathbb{Z}$). Similarly, we identify S with a subset of G_k.

- If three distinct points P_u, P_v, P_w are collinear (*i.e.*, lie on a line), then the line is denoted by $[u, v, w]$. Notice that by the group law of an elliptic curve, we have that three distinct points P_u, P_v, P_w are collinear if and only if $u + v + w \equiv 0 \pmod{9k+3}$. In other words, each line through three different points is of the form $[u, v, w]$ for some pairwise distinct $u, v, w \in G_k$.

- If $[u, v, w]$ is a line, then $-[u, v, w] := [-u, -v, -w]$ is the *inverse line* of $[u, v, w]$. Notice that if $[u, v, w]$ is a line in S (*i.e.*, $u, v, w \in S$), then $-[u, v, w]$ is a line in S with $-[u, v, w] \neq [u, v, w]$, namely the line mirrored at the x-axis.

- For $u \in G_k$, we define $\rho(u) := u + (3k+1)$. Notice that if, for example, $u \in S_0$, then $\rho(u) \in S_1$ and $\rho^2(u) := (\rho \circ \rho)(u) \in S_2$.

- If $[u, v, w]$ is a line, then $\rho[u, v, w] := [\rho u, \rho v, \rho w]$ is the corresponding *rotated line*. Notice that if $[u, v, w]$ is a line in S, then $\rho[u, v, w]$ and $\rho^2[u, v, w]$ are lines in S, where $[u, v, w]$, $\rho[u, v, w]$, and $\rho^2[u, v, w]$ are pairwise distinct (but not necessarily disjoint) lines.

The following fact is an immediate consequence of the preceding definitions.

Fact 2. *Any $(9k_4, 12k_3)$-configuration on the point set S which contains with any line $[u, v, w]$ also the lines $\rho[u, v, w]$ and $\rho^2[u, v, w]$, is an elliptic C_3-symmetric $(9k_4, 12k_3)$-configuration, where C_3 is the cyclic group of order 3. If the configuration contains in addition with any line $[u, v, w]$ also the line $-[u, v, w]$, then it is an elliptic D_3-symmetric configuration.*

So, by Fact 2, to construct an elliptic D_3-symmetric $(9k_4, 12k_3)$ configuration it is enough to find $2k$ lines $[u_i, v_i, w_i]$ such that for $1 \leq i \leq 2k$, the lines $\pm[u_i, v_i, w_i]$,

$\pm\rho[u_i, v_i, w_i]$, and $\pm\rho^2[u_i, v_i, w_i]$ are pairwise distinct. Before we start constructing such lines, we show how we construct lines in S from "pseudo-lines" in S_0:

For any $u, v, w \in S$, let

$$u' := u \pmod{3k+1}, \quad v' := v \pmod{3k+1}, \quad w' := w \pmod{3k+1}.$$

Then $u', v', w' \in S_0$ and if $[u, v, w]$, then $u'+v'+w' \equiv 0 \pmod{3k+1}$. If $u, v, w \in S_0$ are such that $u+v+w \equiv 0 \pmod{3k+1}$, then the triple (u, v, w) is called a *pseudo-line* in S_0. Notice that we do not require that the three points u, v, w of a pseudo-line (u, v, w) are pairwise distinct.

The following lemma shall be crucial in the construction of $(9k_4, 12k_3)$-configurations.

Reduction Lemma 3. *If $u, v, w \in S_0$ are such that (u, v, w) is a pseudo-line, then there are $\bar{u}, \bar{v}, \bar{w} \in S$ such that $u = \bar{u}'$, $v = \bar{v}'$, $w = \bar{w}'$ and $[\bar{u}, \bar{v}, \bar{w}]$ is a line.*

Proof. Let $u, v, w \in S_0$ be such that (u, v, w) be a pseudo-line. Notice that since $u + v + w \equiv 0 \pmod{3k+1}$ and $3 \nmid 3k+1$, at most two of the three points u, v, w can be equal. Without loss of generality assume $u \neq v$. Then, for $\bar{u} := u$, $\bar{v} := v$, and $\bar{w} := w + (6k+2)$, $[\bar{u}, \bar{v}, \bar{w}]$ is a line. *q.e.d.*

In order to construct an elliptic C_3-symmetric $(9k_4, 12k_3)$-configuration, by Reduction Lemma 3 and by rotating the lines with ρ and ρ^2, respectively, it is enough to find a set L of $4k$ pseudo-lines in S_0 such that each point of S_0 belongs to exactly 4 pseudo-lines in L. In order to construct an elliptic D_3-symmetric $(9k_4, 12k_3)$-configuration, we have to make sure in addition that for each pseudo-line $(u, v, w) \in L$, also $(-u, -v, -w) \in L$.

Theorem 4. *For every integer $k \geq 2$ there exists an elliptic D_3-symmetric $(9k_4, 12k_3)$-configuration.*

In the following three sections, we shall construct elliptic $(9k_4, 12k_3)$-configurations for $k \equiv 3 \pmod 4$, for $k \equiv 1 \pmod 4$, and for k even, respectively.

3.1 D_3-symmetric $(9k_4, 12k_3)$-configurations for $k \equiv 3 \pmod 4$

Let $k \geq 3$ be a positive integer with $k \equiv 3 \pmod 4$. Furthermore, let $n_k := 3k+1$ and let

$$m_1 := \frac{k+1}{2}, \qquad m_2 := n_k - m_1,$$

and let

$$t_2 := \begin{cases} \frac{m_1}{2} & \text{if } m_1 \equiv 2 \pmod 4, \\ \frac{n_k+m_1}{2} & \text{otherwise,} \end{cases} \qquad t_1 := n_k - t_2.$$

Notice that since $k \equiv 3 \pmod 4$, we have that $k+1 \equiv 0 \pmod 4$. Hence, we have $m_1 \equiv 0$ or $2 \pmod 4$ and since $n_k \equiv 2 \pmod 4$, we have either $m_1 \equiv 2 \pmod 4$ or $n_k + m_1 \equiv 2 \pmod 4$, which implies that t_1 and t_2 are both odd.

Let $S_0^* := S_0 \cup \{0\}$ and define the following sequence of triples $\langle (a_i, b_i, c_i) : i \in \mathbb{Z} \rangle$ in $S_0^* \times S_0^* \times S_0^*$: Let

$$(a_0, b_0, c_0) := (t_1, 0, t_2)$$

and for all $i \in \mathbb{Z}$ let

$$(a_{i+1}, b_{i+1}, c_{i+1}) := (a_i - 2, b_i + 1, c_i + 1)\,.$$

Then, the sequence has the following properties:

(a) For all $i \in \mathbb{Z}$, $a_i + b_i + c_i \equiv 0 \pmod{n_k}$ and a_i is odd. For the latter, recall that t_1 is odd and that n_k is even.

(b) $(a_{t_1}, b_{t_1}, c_{t_1}) = (t_2, t_1, 0) \pmod{n_k}$. For example, $a_{t_1} = t_1 - 2t_1 = -t_1 \equiv t_2 \pmod{n_k}$.

(c) For all $i \in \mathbb{Z}$, $(a_{i+n_k}, b_{i+n_k}, c_{i+n_k}) = (a_i, b_i, c_i)$, and for all $0 < s < n_k$, we have $\{a_{i+s}, b_{i+s}, c_{i+s}\} \neq \{a_i, b_i, c_i\}$.

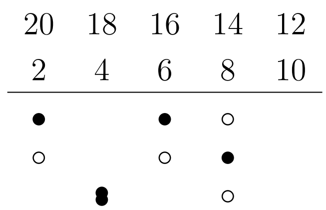
(d) In $\mathbb{Z}/n_k\mathbb{Z}$, for all $s \in \mathbb{Z}$ we have

$$-(a_s, b_s, c_s) = -(t_1 - 2s, s, t_2 + s) = (-t_2 + 2s, -s, t_1 - s) = (a_{t_1 - s}, c_{t_1 - s}, b_{t_1 - s}).$$

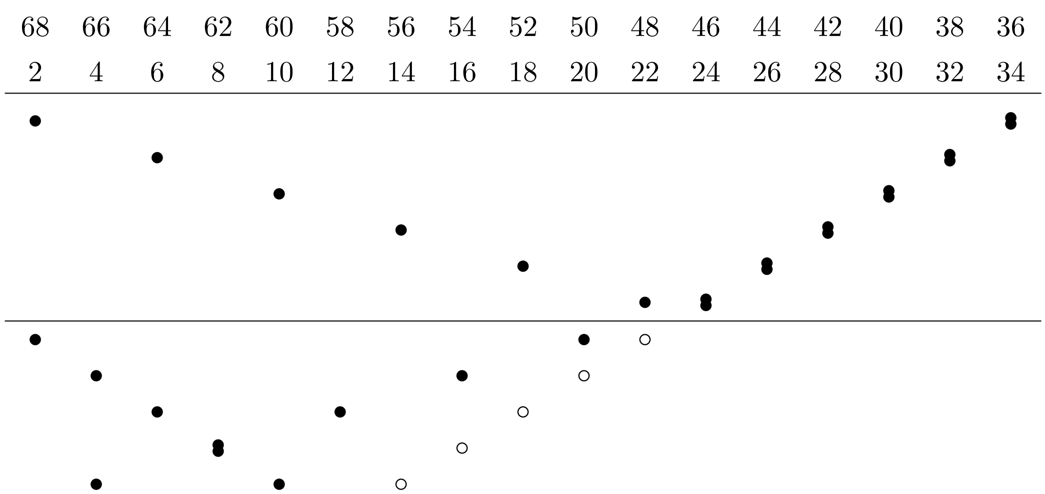
Property (a) shows that every triple in the sequence is a pseudo-line in S_0^*. Property (c) shows that the sequence contains exactly n_k pairwise different pseudo-lines; let L^* be the set of these n_k pseudo-lines. Property (d) shows that a pseudo-line (u, v, w) is in L^* if and only if the pseudo-line $-(u, v, w)$ is in L^*.

Every even number $0 \le \ell < n_k$ appears in exactly 2 pseudo-lines in L^*, and every odd number $0 < \ell < n_k$ appears in exactly 4 pseudo-lines in L^*. Now, we remove the two pseudo-lines $(t_1, 0, t_2)$ and $(t_2, t_1, 0)$ from L^*, and add the two pseudo-lines (m_1, t_1, t_1) and (m_2, t_2, t_2) to L^*; the resulting set of pseudo-lines is denoted L_0. Notice that $(m_2, t_2, t_2) = -(m_1, t_1, t_1)$, that the two pseudo-lines (m_1, t_1, t_1) and (m_2, t_2, t_2) are not in L^*, and that every pseudo-line in L_0 is a pseudo-line in S_0. In L_0, every odd number $0 < \ell < n_k$ appears in exactly 4 pseudo-lines in L_0, and every even number $0 \le \ell < n_k$, except m_1 and m_2, appears in exactly 2 pseudo-lines in L_0, whereas m_1 and m_2 appear in exactly 3 pseudo-lines in L_0.

In order to complete the construction of a $(9k_4, 12k_3)$-configuration, we consider the set T_k consisting of the $\frac{n_k}{2} - 1$ even numbers $2, 4, \ldots n_k - 2$. It remains to find $k-1$ pseudo-lines in S_0 with points in T_k, where every number in T_k except m_1 and m_2 appears in exactly 2 pseudo-lines, whereas m_1 and m_2 appear in exactly 1 pseudo-line. Together with the $n_k = 3k+1$ pseudo-lines of L_0, this gives us $4k$ pseudo-lines, and after extending them to lines of S by Reduction Lemma 3 and by rotating them with ρ and ρ^2, we finally obtain $12k$ lines. For the construction of the remaining $k-1$ pseudo-lines with points in T_k, we give first a "visual argument": We write the points of T_k in two rows, where the first row contains the numbers $n_k - 2$ to $\frac{n_k-2}{2}+1$ in reverse order, and the second row contains the numbers 2 to $\frac{n_k-2}{2}$ in the natural order. Below the numbers of these two rows, we write $\bullet$ and $\circ$ for the three points of the pseudo-lines, where $\bullet$ denotes a number from the second row, and $\circ$ denotes a number from the first row. The following figure gives an example of three pseudo-lines for $k = 7$ (*i.e.*, $n_k = 22$):



The first pseudo-line is $(2, 6, 14)$, the second is $(20, 16, 8) = -(2, 6, 14)$, and the third is $(4, 4, 14)$, where $\bullet$ means twice the same number. Notice that $-(u, v, w)$ is obtained from (u, v, w) by exchanging $\bullet$ and $\circ$. Now, instead of writing both pseudo-lines (u, v, w) and $-(u, v, w)$, we just write the one which the greater number of $\bullet$'s — having in mind that each pseudo-line (u, v, w) represents also the pseudo-line $-(u, v, w)$. This way, we just have to find $\frac{k-1}{2}$ pseudo-lines. The following figure illustrates the 11 pseudo-lines for $k = 23$ (*i.e.*, $n_k = 70$), given in two parts:



First, notice that the pseudo-lines given in the diagram are different from the pseudo-lines constructed above. Furthermore, we see that each point, except the points 12 and 58, appears in exactly 2 pseudo-lines, whereas the points 12 and 58 appear in exactly 1 pseudo-line. Notice that for $k = 23$, $m_1 = \frac{k+1}{2} = 12$ and $m_2 = n_k - m_1 = 58$.

Now, we give a more formal construction of the remaining $\frac{k-1}{2}$ pseudo-lines: Let $\tilde{n}_k := \frac{n_k}{2}$. The $\frac{k+1}{4}$ pseudo-lines in the first part are

$$\big(2+4i, (\tilde{n}_k - 1) - 2i, (\tilde{n}_k - 1) - 2i\big) \quad \text{where } 0 \leq i \leq \tfrac{k-3}{4}.$$

In particular, for $i = 0$ we obtain $(2, \tilde{n}_k - 1, \tilde{n}_k - 1)$, and that for $i = \frac{k-3}{4}$ we obtain $(k-1, k+1, k+1)$ (notice that $2 + 4 \cdot \frac{k-3}{4} = k - 1$ and $(\frac{3k+1}{2} - 1) - 2 \cdot \frac{k-3}{4} = k+1$). Furthermore, the $\frac{k-3}{4}$ pseudo-lines in the second part are

$$\big(2+2i, (k-3) - 4i, -(k-1) + 2i\big) \quad \text{where } 0 \leq i \leq \tfrac{k-7}{4}.$$

In particular, for $i = 0$ we obtain $(2, k-3, -(k-1))$, and that for $i = \frac{k-7}{4}$ we obtain $(\frac{k-3}{2}, 4, -\frac{k+5}{2})$. Notice that $2 + 2 \cdot \frac{k-7}{4} = \frac{k-3}{2}$, $(k-3) - 4 \cdot \frac{k-7}{4} = 4$, and $-(k-1) + 2 \cdot \frac{k-7}{4} = -\frac{k+5}{2}$. Now, since $\frac{k-3}{2} + 2 = m_1$ and $-(\frac{k+5}{2} - 2) = m_2$, we see that the only numbers which appear in exactly one pseudo-line are m_1 and m_2.

Example. We illustrate the construction described above for the parameter $k = 3$. This leads to an elliptic D_3-symmetric $(27_4, 36_3)$-configuration. The underlying group is $\mathbb{Z}_{30}$ on Γ_0. We obtain:

- $k = 3$, $n_k = 10$, $m_1 = 2$, $m_2 = 8$, $t_1 = 9$, $t_2 = 1$.
- The pseudo-lines in L^* are

  $(9,0,1), (7,1,2), (5,2,3), (3,3,4), (1,4,5), (9,5,6), (7,6,7), (5,7,8), (3,8,9), (1,9,0)$.
- Cancel $(9,0,1)$ and $(1,9,0)$, and add $(2,9,9)$ and $(8,1,1)$. This gives us the 10 pseudo-lines of L_0.
- The diagram, which yields the additional $\frac{k-1}{2} = 1$ line consists just of the fist part:

  $$\begin{array}{cc} 8 & 6 \\ 2 & 4 \\ \hline \bullet & \bullet \end{array}$$

  This gives us the lines $(2,4,4)$ and $(8,6,6)$.
- Together with the 10 pseudo-lines in L_0, we have now 12 pseudo-lines which we extend to proper lines in S and rotate them.

Observe that depending on how we extend the pseudo-lines to proper lines, and depending on the choice of the generator of $\mathbb{Z}_{30}$, we obtain different resulting configurations. One version is shown in Figure 2.

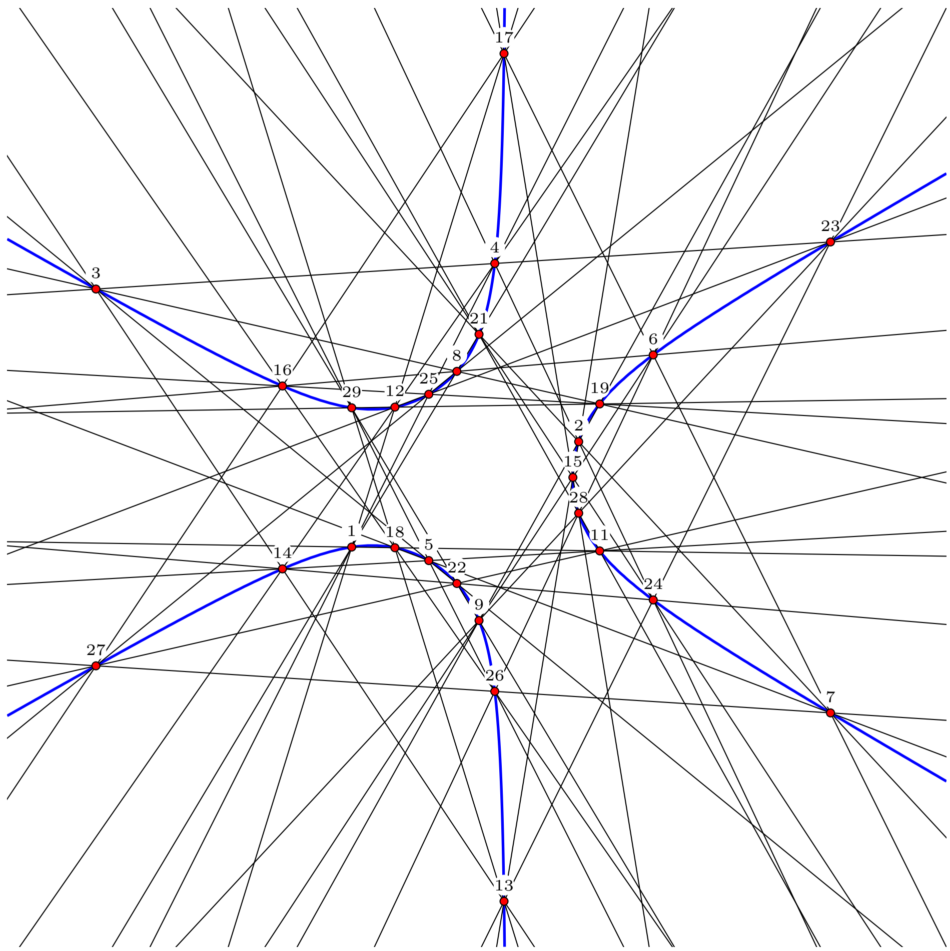


Figure 2: An elliptic D_3-symmetric $(27_4, 36_3)$-configuration.

3.2 D_3-symmetric $(9k_4, 12k_3)$-configurations for $k \equiv 1 \pmod 4$

Let $k \geq 3$ be a positive integer with $k \equiv 1 \pmod 4$. Furthermore, let $n_k := 3k+1$ and let $m := \frac{n_k}{2}$. Notice that since $n_k \equiv 0 \pmod 4$, m is even.

As above, Let $S_0^* := S_0 \cup \{0\}$ and define the following sequence of triples $\langle (a_i, b_i, c_i) : i \in \mathbb{Z} \rangle$ in $S_0^* \times S_0^* \times S_0^*$: Let

$$(a_0, b_0, c_0) := (0, n_k - 1, 1) \quad \text{and} \quad (a_1, b_1, c_1) := (2, n_k - 1, n_k - 1),$$

and for all $i \in \mathbb{Z}$ let

$$(a_{i+2}, b_{i+2}, c_{i+2}) := (a_i + 4, b_i - 2, c_i - 2)\,.$$

Then, the sequence has the following properties:

(a) For all $i \in \mathbb{Z}$, $a_i + b_i + c_i \equiv 0 \pmod{n_k}$, a_i is even, and b_i and c_i are both odd.

(b) $(a_m, b_m, c_m) = (0, m-1, m+1)$.

(c) For all $i \in \mathbb{Z}$, $(a_{i+n_k}, b_{i+n_k}, c_{i+n_k}) = (a_i, b_i, c_i)$, and for all all $0 < s < n_k$, $\{a_{i+s}, b_{i+s}, c_{i+s}\} \neq \{a_i, b_i, c_i\}$.

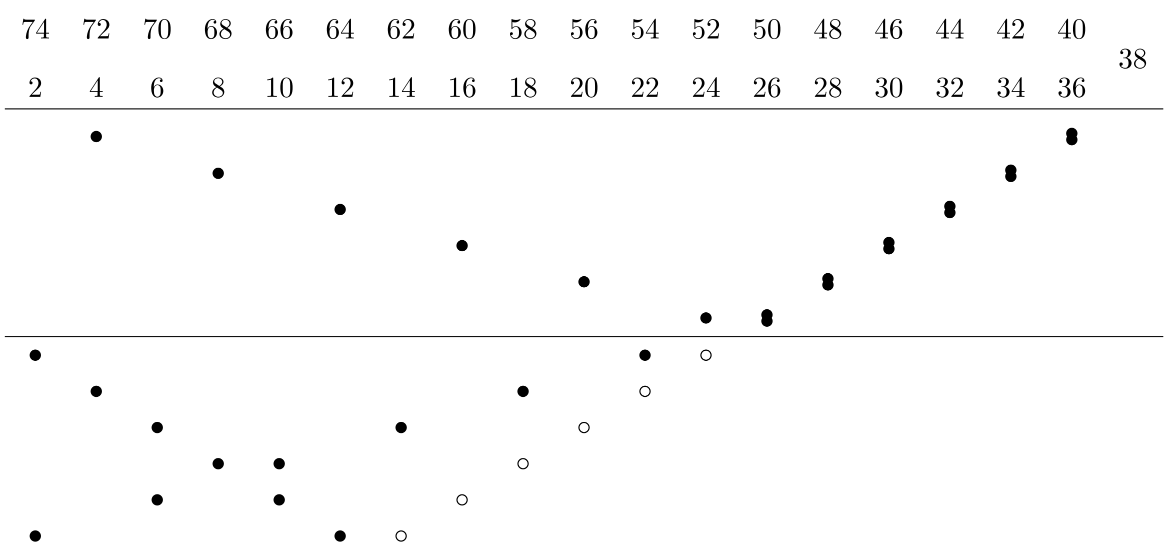
(d) In $\mathbb{Z}/n_k\mathbb{Z}$, for all $s \in \mathbb{Z}$ we have $-(a_s, b_s, c_s) = (a_{-s}, b_{-s}, c_{-s})$.

Property (a) shows that every triple in the sequence is a pseudo-line in S_0^*. Property (c) shows that the sequence contains exactly n_k pairwise different pseudo-lines; let L^* be the set of these n_k pseudo-lines. Property (d) shows that a pseudo-line (u, v, w) is in L^* if and only if the pseudo-line $-(u, v, w)$ is in L^*.

Every even number $0 \leq \ell < n_k$ appears in exactly 2 pseudo-lines in L^*, and every odd number $0 < \ell < n_k$ appears in exactly 4 pseudo-lines in L^*. Now, we remove the two pseudo-lines $(0, n_k - 1, 1)$ and $(0, m-1, m+1)$ from L^*, and add the two pseudo-lines $(m, n_k - 1, m+1)$ and $(m, 1, m-1)$ to L^*; the resulting set of pseudo-lines is denoted L_0. Notice that $(m, n_k - 1, m+1) = -(m, 1, m-1)$, that the two pseudo-lines $(m, n_k - 1, m+1)$ and $(m, 1, m-1)$ are not in L^*, and that every pseudo-line in L_0 is a pseudo-line in S_0. In L_0, every odd number $0 < \ell < n_k$ appears in exactly 4 pseudo-lines in L_0, and every even number $0 \leq \ell < n_k$, except m, appears in exactly 2 pseudo-lines in L_0, whereas m appears in exactly 4 pseudo-lines in L_0.

In order to complete the construction of a $(9k_4, 12k_3)$-configuration, we consider the set T_k consisting of the $\frac{n_k}{2} - 1$ even numbers $2, 4, \ldots n_k - 2$. It remains to find $k - 1$ pseudo-lines in S_0 with points in T_k, where every number in T_k except m appears in exactly 2 pseudo-lines, whereas m does not appear in any pseudo-line.

For the construction of the remaining $k-1$ pseudo-lines with points in T_k, we give again first a "visual argument": As above, we write just the pseudo-line with the greater number of $\bullet$'s — having in mind that each pseudo-line (u, v, w) represents also the pseudo-line $-(u, v, w)$. This way, we just have to find $\frac{k-1}{2}$ pseudo-lines. The following figure illustrates the 12 pseudo-lines for $k = 25$ (*i.e.*, $n_k = 76$), given in two parts:



First, notice that the pseudo-lines given in the diagram are different to the pseudo-lines constructed above. Furthermore, we see that each point, except the point 38, appears in exactly 2 pseudo-lines, whereas the point 38 does not appear in a pseudo-line. Notice that for $k = 25$, $m = 38$.

Now, we give a more formal construction of the remaining $\frac{k-1}{2}$ pseudo-lines: The $\frac{k-1}{4}$ pseudo-lines in the first part are

$$\big(4+4i, (m-2)-2i, (m-2)-2i\big) \quad \text{where } 0 \leq i \leq \tfrac{k-5}{4}.$$

In particular, for $i = 0$ we obtain $(4, m-2, m-2)$, and that for $i = \frac{k-5}{4}$ we obtain $(k-1, k+1, k+1)$ (recall that $m = \frac{3k+1}{2}$). Furthermore, the $\frac{k-1}{4}$ pseudo-lines in the second part are

$$\big(2+2i, (k-3)-4i, -(k-1)+2i)\big) \quad \text{where } 0 \leq i \leq \tfrac{k-5}{4}.$$

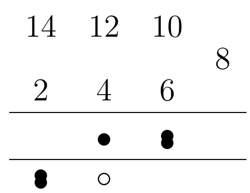
In particular, for $i = 0$ we obtain $(2, k-3, -(k-1))$, and that for $i = \frac{k-5}{4}$ we obtain $(\frac{k-1}{2}, 2, -\frac{k+3}{2})$. Notice that the only number which does not appear in a pseudo-line is m, as required.

Example. We illustrate this construction for the parameter $k = 5$. This leads to an elliptic D_3-symmetric $(45_4, 60_3)$-configuration. The construction gives the following:

- $k = 5$, $n_k = 16$, $m = 8$.
- The pseudo-lines in L^* are:

  $(0, 15, 1), (2, 15, 15), (4, 13, 15), (6, 13, 13), (8, 11, 13), (10, 11, 11), (12, 9, 11),$
  $(14, 9, 9), (0, 7, 9), (2, 7, 7), (4, 5, 7), (6, 5, 5), (8, 3, 5), (10, 3, 3), (12, 1, 3), (14, 1, 1)$

- Cancel $(0, 15, 1)$ and $(0, 7, 9)$, and add $(8, 15, 9)$ and $(8, 1, 7)$. This gives us the 16 pseudo-lines of L_0.
- The diagram, which gives us additional $\frac{k-1}{2} = 2$ lines consists of just two lines, one line in each part:

  This gives us the $k - 1 = 4$ lines $(4, 6, 6)$, $(12, 10, 10)$, $(2, 2, 12)$, $(14, 14, 4)$.

- Together with the 16 pseudo-lines in L_0, we have now 20 pseudo-lines which we extend to proper lines in S and rotate them.

Again, depending on how we extend the pseudo-lines to proper lines, and depending on the choice of the generator of $\mathbb{Z}_{48}$, we obtain different resulting configurations. One version is shown in Figure 3.

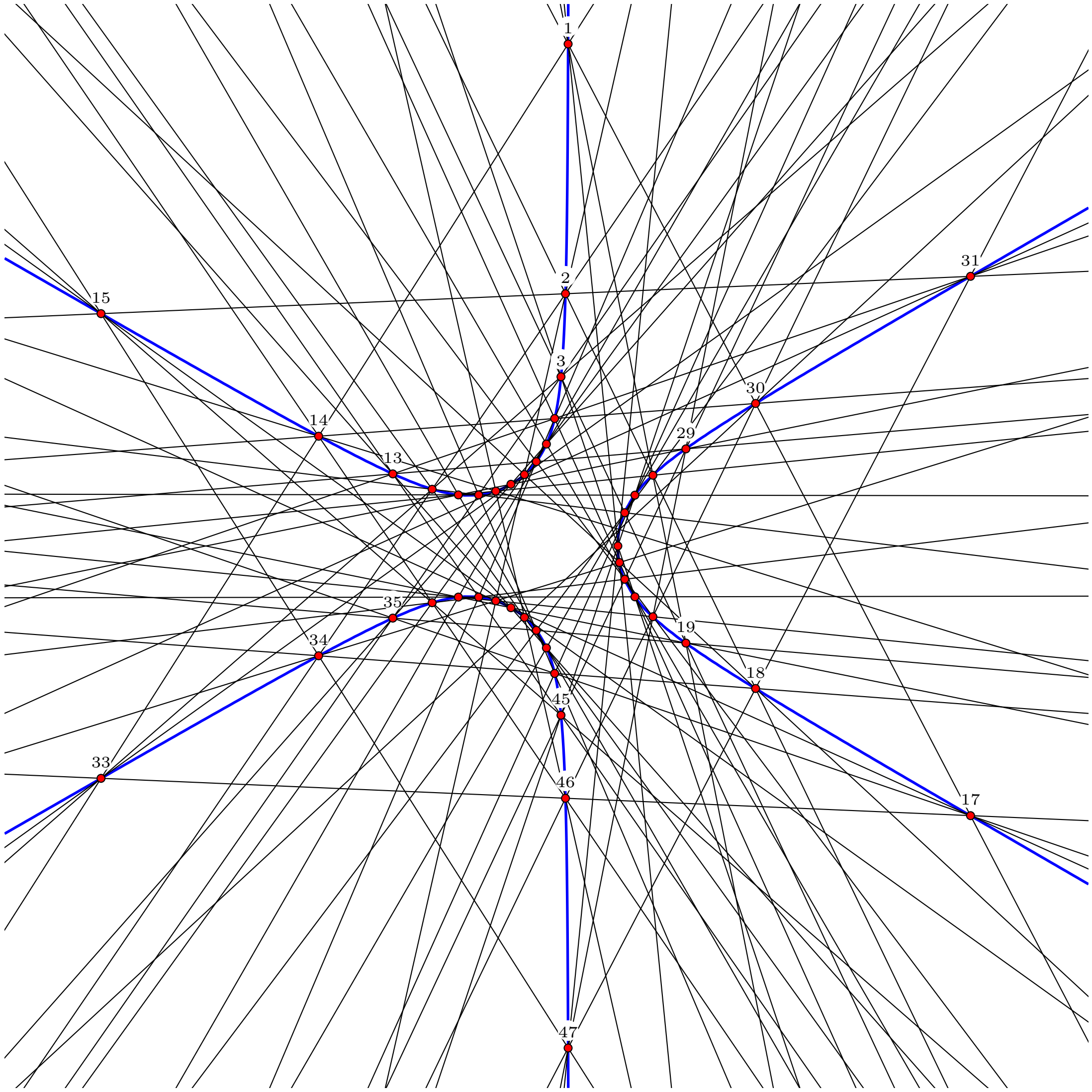


Figure 3: An elliptic D_3-symmetric $(45_4, 60_3)$-configuration. For this figure we have chosen the generator 1 in $\mathbb{Z}_{48}$.

3.3 D_3-symmetric $(9k_4, 12k_3)$-configurations for k even

Let $k \geq 2$ be an even integer and let $n_k := 3k + 1$. Notice that n_k is odd.

As above, Let $S_0^* := S_0 \cup \{0\}$ and define the following sequence of triples $\langle (a_i, b_i, c_i) : i \in \mathbb{Z} \rangle$ in $S_0^* \times S_0^* \times S_0^*$: Let $(a_0, b_0, c_0) := (0, 0, 0)$ and for all $i \in \mathbb{Z}$ let

$$(a_{i+1}, b_{i+1}, c_{i+1}) := (a_i - 2, b_i + 1, c_i + 1) .$$

Then, the sequence has the following properties:

(a) For all $i \in \mathbb{Z}$, $a_i + b_i + c_i \equiv 0 \pmod{n_k}$, and $b_i = c_i$.

(b) For $t := \frac{3k}{2}$ we have $(a_t, b_t, c_t) = (1, t, t)$.

(c) For all $i \in \mathbb{Z}$, $(a_{i+n_k}, b_{i+n_k}, c_{i+n_k}) = (a_i, b_i, c_i)$, and for all all $0 < s < n_k$, $\{a_{i+s}, b_{i+s}, c_{i+s}\} \neq \{a_i, b_i, c_i\}$.

(c) In $\mathbb{Z}/n_k\mathbb{Z}$, for $t := \frac{3k}{2}$ and for all $s \in \mathbb{Z}$ we have

$$-(a_{t+s}, b_s, c_s) = (a_{t-s+1}, b_{t-s+1}, c_{t-s+1}).$$

Property (a) shows that every triple in the sequence is a pseudo-line in S_0^*. Property (c) shows that the sequence contains exactly n_k pairwise different pseudo-lines, including the pseudo-line $(0, 0, 0)$. Now, we remove the pseudo-line $(0, 0, 0)$ and let L_0 be the set of the remaining $3k$ pseudo-lines. Property (d) shows that a pseudo-line (u, v, w) is in L_0 if and only if the pseudo-line $-(u, v, w)$ is in L_0. Furthermore, notice that every number $0 < \ell < n_k$ appears in exactly 3 pseudo-lines in L_0.

For the construction of the remaining k pseudo-lines in S_0, we will again visualise the argument. As above, we write just the pseudo-line with the greater number of $\bullet$'s — having in mind that each pseudo-line (u, v, w) represents also the pseudo-line $-(u, v, w)$. This way, we just have to find $\frac{k}{2}$ pseudo-lines. In order to show the structure of the construction, we omit the least point of a pseudo-line and write the number of the least point as an index to the the other two points of the pseudo-line. For example, for $k = 4$ (*i.e.*, $n_k = 13$) and the pseudo-line $(1, 3, 9)$ we shall write:

$$\begin{array}{cccccc} 12 & 11 & 10 & 9 & 8 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \hline & & \bullet_1 & \circ_1 & & \end{array} \qquad \text{instead of} \qquad \begin{array}{cccccc} 12 & 11 & 10 & 9 & 8 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \bullet & & \bullet & \circ & & \end{array}$$

This way, we can write different pseudo-lines in the same row without ambiguity. For example, for $k = 8$ (*i.e.*, $n_k = 25$) the following diagram represents the three pseudo-lines $(1, 7, 17)$, $(2, 9, 14)$, and $(4, 6, 15)$:

$$\begin{array}{cccccccccccc} 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline & & & & & \bullet_4 & \bullet_1 & \circ_1 & \bullet_2 & \circ_4 & \circ_2 & \end{array}$$

We first consider the cases when $k = 2, 4, 6, 8, 10, 12, 14, 16, 18$, and then we consider the cases when $k \geq 22$, where we shall consider the four cases $k \equiv 0, 2, 4, 6 \pmod 8$ separately.

The following diagrams show the $\frac{k}{2}$ pseudo-lines for $k = 2, 4, 6, 8, 10, 12, 14, 16, 18, 20$ (where we do not write the points which appear as indices):

5	4
2	3
$\bullet_1$	$\circ_1$

$k = 2$

10	9	8	7
3	4	5	6
$\bullet_1$	$\circ_1$	$\bullet_2$	$\bullet_2$

$k = 4$

15	14	13	12	11	10
4	5	6	7	8	9
$\bullet_1$	$\circ_1$	$\bullet_2$	$\bullet_3$	$\circ_2$	$\bullet_3$

$k = 6$

20	19	18	17	16	15	14	13
5	6	7	8	9	10	11	12
$\bullet_4$	$\bullet_2$		$\circ_2$	$\circ_4$			
		$\bullet_3$			$\circ_3$	$\bullet_1$	$\circ_1$

$k = 8$

25	24	23	22	21	20	19	18	17	16
6	7	8	9	10	11	12	13	14	15
$\bullet_4$				$\circ_4$	$\bullet_2$		$\circ_2$		
	$\bullet_1$	$\circ_1$	$\bullet_5$			$\bullet_3$		$\circ_5$	$\circ_3$

$k = 10$

30	29	28	27	26	25	24	23	22	21	20	19
7	8	9	10	11	12	13	14	15	16	17	18
				$\bullet_6$	$\bullet_4$	$\bullet_2$		$\circ_2$	$\circ_4$	$\circ_6$	
$\bullet_3$	$\bullet_1$	$\circ_1$	$\circ_3$				$\bullet_5$				$\bullet_5$

$k = 12$

35	34	33	32	31	30	29	28	27	26	25	24	23	22
8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\bullet_6$	$\bullet_4$	$\bullet_2$		$\circ_2$	$\circ_4$	$\circ_6$							
			$\bullet_7$				$\bullet_5$	$\bullet_1$	$\circ_1$	$\circ_7$	$\bullet_3$	$\circ_5$	$\bullet_3$

$k = 14$

40	39	38	37	36	35	34	33	32	31	30	29	28	27	26	25
9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$\bullet_8$	$\bullet_6$	$\bullet_4$	$\bullet_2$		$\circ_2$	$\circ_4$	$\circ_6$	$\circ_8$							
				$\bullet_7$					$\bullet_3$	$\bullet_5$	$\circ_7$	$\circ_3$	$\bullet_1$	$\circ_1$	$\circ_5$

$k = 16$

45	44	43	42	41	40	39	38	37	36	35	34	33	32	31	30	29	28
10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$\bullet_8$	$\bullet_6$	$\bullet_4$	$\bullet_2$		$\circ_2$	$\circ_4$	$\circ_6$	$\circ_8$									
				$\bullet_9$					$\bullet_3$	$\bullet_7$	$\bullet_5$	$\circ_3$	$\circ_9$	$\bullet_1$	$\circ_1$	$\circ_5$	$\circ_7$

$k = 18$

50	49	48	47	46	45	44	43	42	41	40	39	38	37	36	35	34	33	32	31
11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\bullet_{10}$	$\bullet_8$	$\bullet_6$	$\bullet_4$	$\bullet_2$		$\circ_2$	$\circ_4$	$\circ_6$	$\circ_8$	$\circ_{10}$									
					$\bullet_9$						$\bullet_5$	$\bullet_3$	$\bullet_7$	$\circ_9$	$\circ_3$	$\circ_5$	$\bullet_1$	$\circ_1$	$\bullet_7$

$k = 20$

Notice that in the diagrams above, in the case when $k \equiv 4, 6 \pmod 8$, there is always a single pseudo-line which contains just points from the second row. In fact, this will always be the case. Another feature of the diagrams above is that all the numbers $1, \ldots, \frac{k}{2}$ appear as indices — also this will always be the case.

As mentioned above, for $k \geq 20$ we shall consider the four cases $k \equiv 0, 2, 4, 6 \pmod 8$ separately. However, the structure of the pseudo-lines consisting only of even numbers is always the same. This structure is illustrated by the following diagram in which we omit the first row, since it can be easily computed from the second row. In the diagram, u denotes the largest *even* number which is less than or equal to $\frac{k}{2}$ (*i.e.*, u is either $\frac{k}{2}$ or $\frac{k}{2} - 1$), $M := \frac{k+u+2}{2}$, and $N := \frac{k}{2} + 1$:

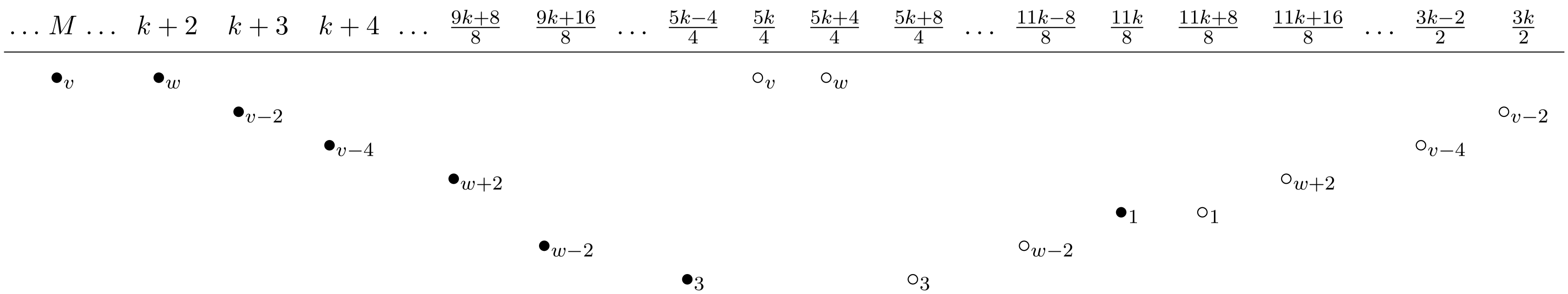
$\ldots$	N	$N+1$	$\ldots$	$M-2$	$M-1$	M	$M+1$	$M+2$	$\ldots$	$N+u-1$	$N+u$	$\ldots$
					$\bullet_2$		$\circ_2$					
				$\bullet_4$				$\circ_4$				
			$\ldots$						$\ldots$			
		$\bullet_{u-2}$								$\circ_{u-2}$		
	$\bullet_u$										$\circ_u$	

We shall call these $\frac{u}{2}$ pseudo-lines the *even block*. Notice that the structure of the even block already appears for $k = 14, 16, 18$.

In order to complete the proof of Theorem 4, we have to construct the remaining $\frac{k}{2} - \frac{u}{2}$ pseudo-lines which consist only of *odd* numbers, the so-called *odd block*. The following four diagrams show the structure of these odd blocks for $k \geq 22$.

The structure of the odd block for $k \geq 24$ and $k \equiv 0 \pmod 8$

Let now $k \equiv 0 \pmod 8$ with $k \geq 24$. Then $u = \frac{k}{2}$, $M = \frac{3k}{4} + 1$, $N = \frac{k}{2} + 1$, and $N + u = k + 1$. Furthermore, let $v := \frac{k}{2} - 1$ and $w := \frac{k}{4} - 1$; then $M + v = \frac{5k}{4}$. Notice that v and w are both odd. The following diagram illustrates the construction of the odd block:



Notice that the odd block fits well with the even block: For example, the number M, which was missing in the even block, appears in the pseudo-line $(v, M, -(M + v)$ (recall that $M + v = \frac{5k}{4}$). Furthermore, the number $N + u + 1 = k + 2$, which is the least number which is bigger than the maximum of the numbers in the even block, appears in the pseudo-line $(w, k{+}2, -(\frac{5k}{4}{+}1))$. The other numbers of the odd block are covered by the pseudo-lines with least number $v{-}2, v{-}4, \ldots, w + 2, w - 2, \ldots, 3, 1$, where the pseudo-line $(1, \frac{11k}{8}, -\frac{11k+8}{8})$ covers the gap between the pseudo-lines $(w + 2, \frac{9k+8}{8}, -\frac{11k+16}{8})$ and $(w - 2, \frac{9k+16}{8}, -\frac{11k-8}{8})$.

The structure of the odd block for $k \geq 26$ and $k \equiv 2 \pmod 8$

For $k \equiv 2 \pmod 8$ with $k \geq 26$ let $u = \frac{k}{2} - 1$, $M = \frac{3k+2}{4}$, $N = \frac{k}{2} + 1$, and $N + u = k$. Furthermore, let $v := \frac{k}{2}$ and $w := \frac{k-6}{4}$; then $M + v = \frac{5k+2}{4}$. The following diagram illustrates the construction of the odd block:

$\ldots M \ldots$	$k+1$	$k+2$	$k+3$	$\ldots$	$\frac{9k+6}{8}$	$\frac{9k+14}{8}$	$\ldots$	$\frac{5k-6}{4}$	$\frac{5k-2}{4}$	$\frac{5k+2}{4}$	$\frac{5k+6}{4}$	$\ldots$	$\frac{11k-14}{8}$	$\frac{11k-6}{8}$	$\frac{11k+2}{8}$	$\frac{11k+10}{8}$	$\ldots$	$\frac{3k-2}{2}$	$\frac{3k}{2}$
$\bullet_{v}$	$\bullet_{w}$	$\bullet_{v-2}$	$\bullet_{v-4}$		$\bullet_{w+2}$	$\bullet_{w-2}$		$\bullet_{3}$	$\circ_{w}$	$\circ_{v}$	$\circ_{3}$		$\circ_{w-2}$	$\bullet_{1}$	$\circ_{1}$	$\circ_{w+2}$		$\circ_{v-4}$	$\circ_{v-2}$

The structure of the odd block for $k \geq 20$ and $k \equiv 4 \pmod 8$

For $k \equiv 4 \pmod 8$ with $k \geq 20$ let $u = \frac{k}{2}$, $M = \frac{3k+4}{4}$, $N = \frac{k}{2} + 1$, and $N + u = k + 1$. Furthermore, let $v := \frac{k}{2} - 1$ and $w := \frac{k+8}{4}$; then $M + v = \frac{5k}{4}$. The following diagram illustrates the construction of the odd block:

$\ldots M \ldots$	$k+2$	$k+3$	$\ldots$	$\frac{9k-12}{8}$	$\frac{9k-4}{8}$	$\ldots$	$\frac{5k-8}{4}$	$\frac{5k-4}{4}$	$\frac{5k}{4}$	$\frac{5k+4}{4}$	$\ldots$	$\frac{11k-4}{8}$	$\frac{11k+4}{8}$	$\frac{11k+12}{8}$	$\frac{11k+20}{8}$	$\ldots$	$\frac{3k-4}{2}$	$\frac{3k-2}{2}$	$\frac{3k}{2}$
$\bullet_{v}$	$\bullet_{v-2}$	$\bullet_{v-4}$		$\bullet_{w+2}$	$\bullet_{w-2}$		$\bullet_{3}$	$\bullet_{w}$	$\circ_{v}$	$\circ_{3}$		$\circ_{w-2}$	$\bullet_{1}$	$\circ_{1}$	$\circ_{w+2}$		$\circ_{v-4}$	$\circ_{v-2}$	$\bullet_{w}$

The structure of the odd block for $k \geq 22$ and $k \equiv 6 \pmod 8$

For $k \equiv 6 \pmod 8$ with $k \geq 22$ let $u = \frac{k}{2} - 1$, $M = \frac{3k+2}{4}$, $N = \frac{k}{2} + 1$, and $N + u = k$. Furthermore, let $v := \frac{k}{2}$ and $w := \frac{k-6}{4}$; then $M + v = \frac{5k+2}{4}$. The following diagram illustrates the construction of the odd block:

$\ldots M \ldots$	$k+1$	$k+2$	$\ldots$	$\frac{9k-6}{8}$	$\frac{9k+2}{8}$	$\frac{9k+10}{8}$	$\frac{9k+18}{8}$	$\ldots$	$\frac{5k-2}{4}$	$\frac{5k+2}{4}$	$\frac{5k+6}{4}$	$\frac{5k+10}{4}$	$\ldots$	$\frac{11k-2}{8}$	$\frac{11k+6}{8}$	$\ldots$	$\frac{3k-4}{2}$	$\frac{3k-2}{2}$	$\frac{3k}{2}$
$\bullet_{v}$	$\bullet_{v-2}$	$\bullet_{v-4}$		$\bullet_{w+2}$	$\bullet_{1}$	$\circ_{1}$	$\bullet_{w-2}$		$\bullet_{3}$	$\circ_{v}$	$\bullet_{w}$	$\circ_{3}$		$\circ_{w-2}$	$\circ_{w+2}$		$\circ_{v-4}$	$\circ_{v-2}$	$\bullet_{w}$

Example. We consider the case $k = 2$ which yields an elliptic D_3-symmetric $(18_4, 24_3)$-configuration. Figure 4 shows one realization of the resulting configurations.

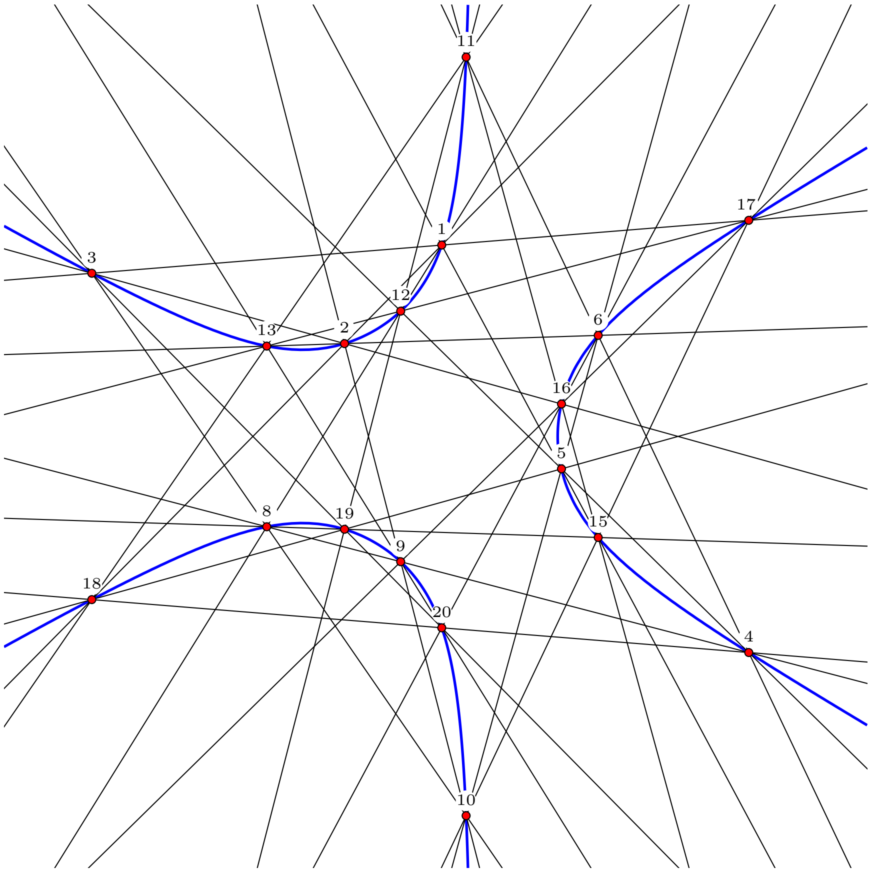


Figure 4: An elliptic D_3-symmetric $(18_4, 24_3)$-configuration.

4 On elliptic $(3r_4, 4r_3)$-configurations

In order to obtain an elliptic D_3-symmetric $(9k_4, 12k_3)$-configuration, it was sufficient to construct $4k$ pseudo-lines in the $3k$-element set S_0. Thus, if all the pseudo-lines we constructed were proper lines, then we would have an elliptic $(3k_4, 4k_3)$-configuration — but this is in general not the case.

However, there is a simple algorithm which gives us elliptic $(3r_4, 4r_3)$-configurations for infinitely many values of r. The algorithm is given in the proof of the following

Proposition 5. *For every prime $p > 7$, there is an elliptic $((p-1)_3)$-configuration and for every prime $p > 7$ with $3r = p - 1$ (for some r), there is an elliptic $(3r_4, 4r_3)$-configuration.*

Proof. Let $p > 7$ be a prime, let Γ_0 be an elliptic curve, and let P be a point on Γ_0 of order $p-1$. Furthermore, let $\mathbb{F}_p$ be the Galois field of order p. Similar as above, we shall construct the elliptic configurations in $\mathbb{F}_p \setminus \{0\}$.

First recall that for any prime p, the multiplicative group $\mathbb{F}_p$ is cyclic, *i.e.*, there exists a generator $g \in \mathbb{F}_p$ such that $\operatorname{ord}(g) = p-1$. Before we start the construction, let us prove the following

Claim. If $p > 7$ is a prime, then the multiplicative group of $\mathbb{F}_p$ has a generator g such that $g \not\equiv -2, \frac{p-1}{2} \pmod{p}$.

Proof of Claim. If $\mathbb{F}_p$ has a generator g such that $g \not\equiv -2, \frac{p-1}{2}$, then we are done. Now, assume that $g = \frac{p-1}{2}$ is a generator. Then, for any n with $1 < n < p-1$ and $(n, p-1) = 1$, g^n is also a generator. So, if we find two distinct n, m with $1 < n, m < p-1$ and $(n, p-1) = 1 = (m, p-1)$, then g, g^n, and g^m are pairwise distinct generators and we have found a generator which satisfies the conditions in the Claim. It remains to show that for every prime $p > 7$ there are distinct n, m with $1 < n, m < p-1$ such that $(n, p-1) = 1 = (m, p-1)$, which is is obviously the case.

Let now $p > 7$ be a prime and let g be a generator of the multiplicative group of $\mathbb{F}_p$ with $g \not\equiv -2, \frac{p-1}{2} \pmod{p}$ and let

$$L_0 := \Big\{ \big(g^n, g^{n+1}, -(g^n + g^{n+1})\big) : 0 \leq n < p-1 \Big\}.$$

Then L_0 is a set of $p-1$ lines in $\mathbb{F}_p \setminus \{0\}$. To see this, notice that by the properties of g, for all n we have $g^n \neq g^{n+1}$ and that $-(g^n + g^{n+1}) \in \{g^n, g^{n+1}\}$ would imply that $g \equiv \frac{p-1}{2} \pmod{p}$ or $g \equiv -2 \pmod{p}$.

Now, with the $p-1$ lines in $\mathbb{F}_p \setminus \{0\}$ and the point P on Γ_0 of order $p-1$, we can easily construct a $((p-1)_3)$-configuration with all its points on Γ_0.

Let us now assume that in addition to $p > 7$ we have that $p - 1 = 3r$ for some $r \geq 4$, and let again g be a generator of the multiplicative group of $\mathbb{F}_p$ with $g \not\equiv -2, \frac{p-1}{2} \pmod{p}$. Let $x := g^r$ and let $y := 1 + x + x^2$. Then, since $x^3 = 1$, we have $x(1 + x + x^2) = x$, which implies that $x \equiv 0 \pmod{p}$ or $1 + x + x^2 \equiv 0 \pmod{p}$. Since the former is impossible (recall that g is a generator of the multiplicative group of $\mathbb{F}_p$), we have that $1 + x + x^2 \equiv 0 \pmod{p}$, and since $1, x, x^2$ are pairwise distinct, this implies that $(1 + x + x^2)$ is a line in $\mathbb{F}_p$. Consequently,

$$L_1 := \big\{ a \cdot \big(1, x, x^2\big)\big) : a \in \mathbb{F}_p \setminus \{0\} \big\}$$

is an r-element set of lines in $\mathbb{F}_p$ which is disjoint from L_0. To see this, notice that no element of L_0 is of the form $a \cdot (1, x, x^2))$ for some $a \in \mathbb{F}_p \setminus \{0\}$ and for all $a, b \in \mathbb{F}_p \setminus \{0\}$, if $\{a, ax, ax^2\} \cap \{b, bx, bx^2\} \neq \emptyset$ then $\{a, ax, ax^2\} = \{b, bx, bx^2\}$. Thus, $L_0 \cup L_1$ is a $4r$-element set of lines in $\mathbb{F}_p \setminus \{0\}$ and together with the point P on Γ_0 of order $p-1$, we can easily construct a $(3r_4, 4r_3)$-configuration with all its points on Γ_0. *q.e.d.*

Examples. We illustrate the construction of the previous proof for the cases $r = 6$, *i.e.*, we deal with the prime $p = 3r + 1 = 19$. The set L_0 contains the lines

$$(1,3,15),(1,4,14),(1,5,13),(2,6,11),(2,7,10),(2,8,9),\\ (3,4,12),(3,7,9),(4,16,18),(5,6,8),(5,15,18),(6,14,18),\\ (7,15,16),(8,13,17),(9,12,17),(10,11,17),(10,12,16),(11,13,14).$$

The set L_1 adds the lines

$$(1,7,11),(2,3,14),(4,6,9),(5,16,17),(8,12,18),(10,13,15).$$

The resulting elliptic D_1-symmetric $(18_4, 24_3)$-configuration is shown in Figure 5 (compare to Figure 4). Here, we have chosen the generator 1 in the multiplicative group of $\mathbb{F}_{19}$.

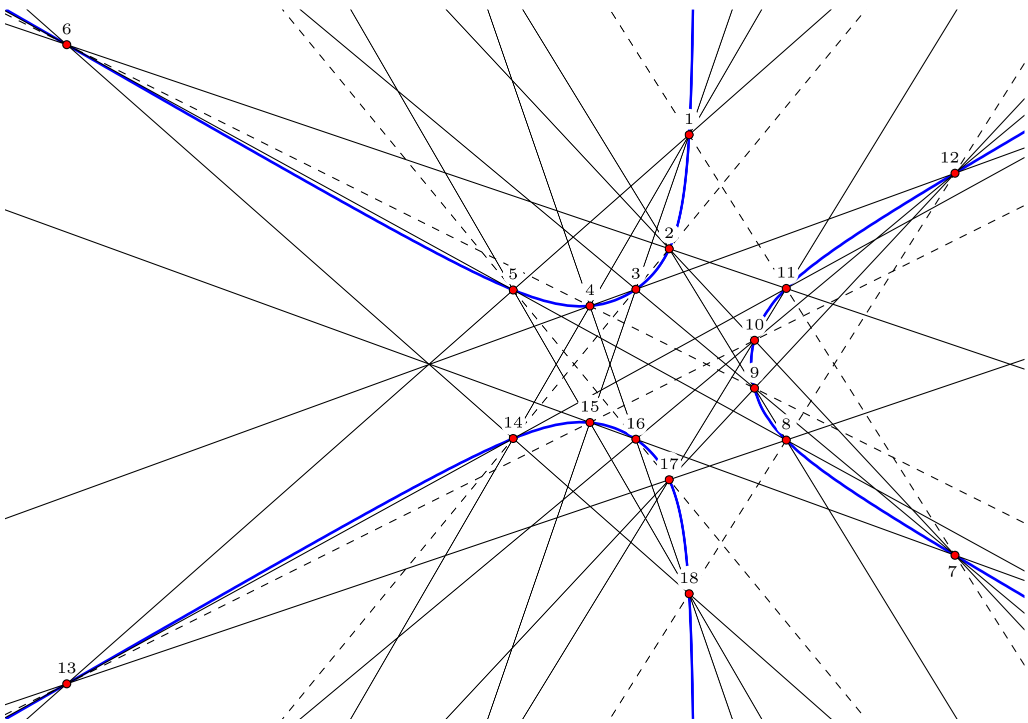


Figure 5: The solid and the dashed lines form an elliptic D_1-symmetric $(18_4, 24_3)$-configuration derived from $\mathbb{Z}/19\mathbb{Z}$. The solid lines in the set L_0 alone are an elliptic D_1-symmetric (18_3)-configuration.

We also add the case $k = 10$, *i.e.*, the prime $p = 3r + 1 = 31$. We omit the list of points and refer directly to Figure 6.

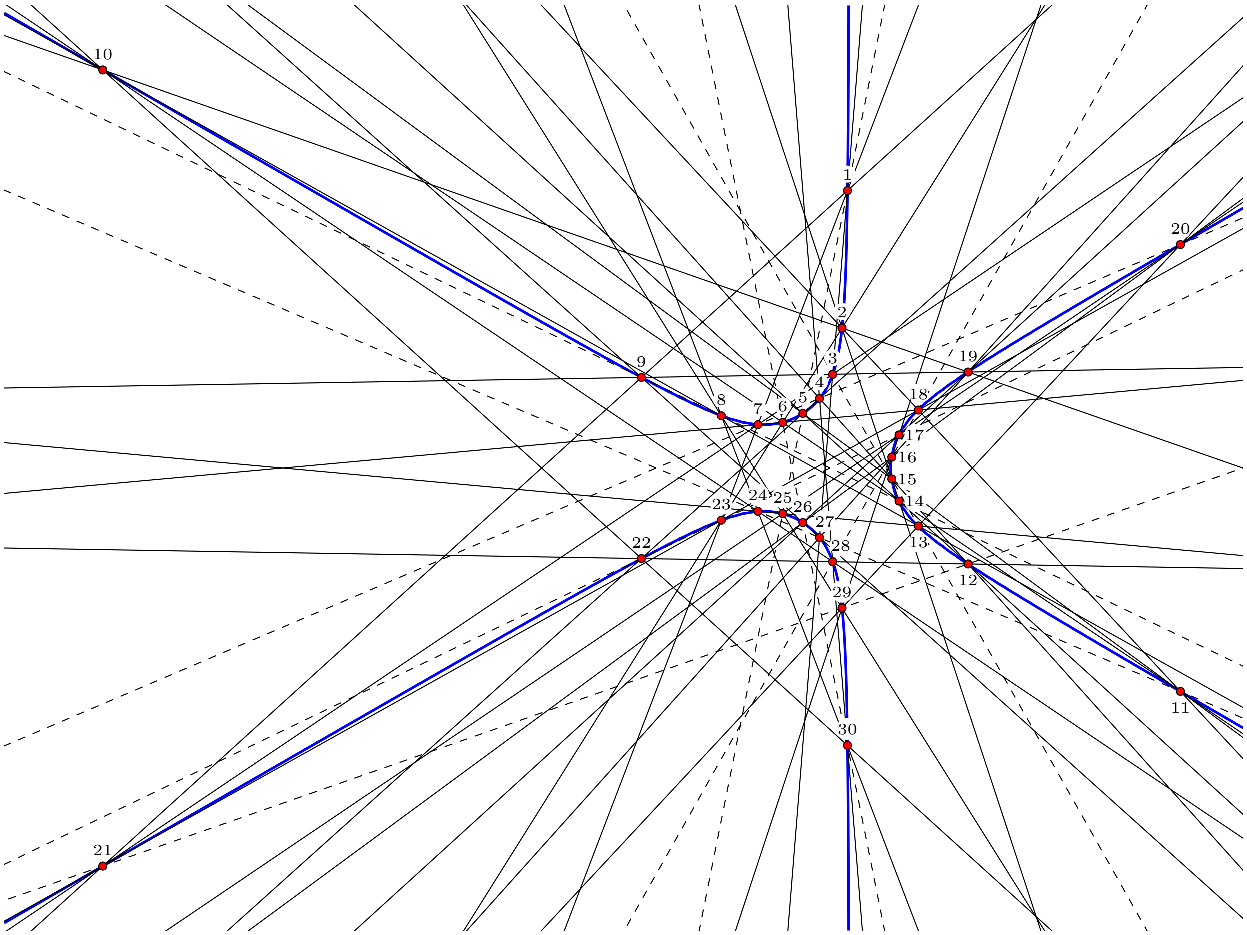


Figure 6: The solid and the dashed lines form an elliptic D_1-symmetric $(30_4, 40_3)$-configuration derived from $\mathbb{Z}/31\mathbb{Z}$, the solid lines alone are an elliptic D_1-symmetric (30_3)-configuration. Observe that a $(30_4, 40_3)$-configuration cannot be realized by the methods from Section 3.

5 Elliptic configurations resulting from groups of the form $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}$

We conclude this paper by presenting some $(3r_4, 4r_3)$-configurations which are derived from groups of the form $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}$ by similar methods. In Figure 7 we realize the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ on an elliptic curve consisting of two components. There are 15 real points and the point $\mathscr{O}$ at infinity $(0, 1, 0)$. Using all real points the result is an elliptic D_1-symmetric $(15_4, 20_3)$-configuration. Notice that such a configuration cannot be constructed by the methods presented in Section 3 and Section 4.

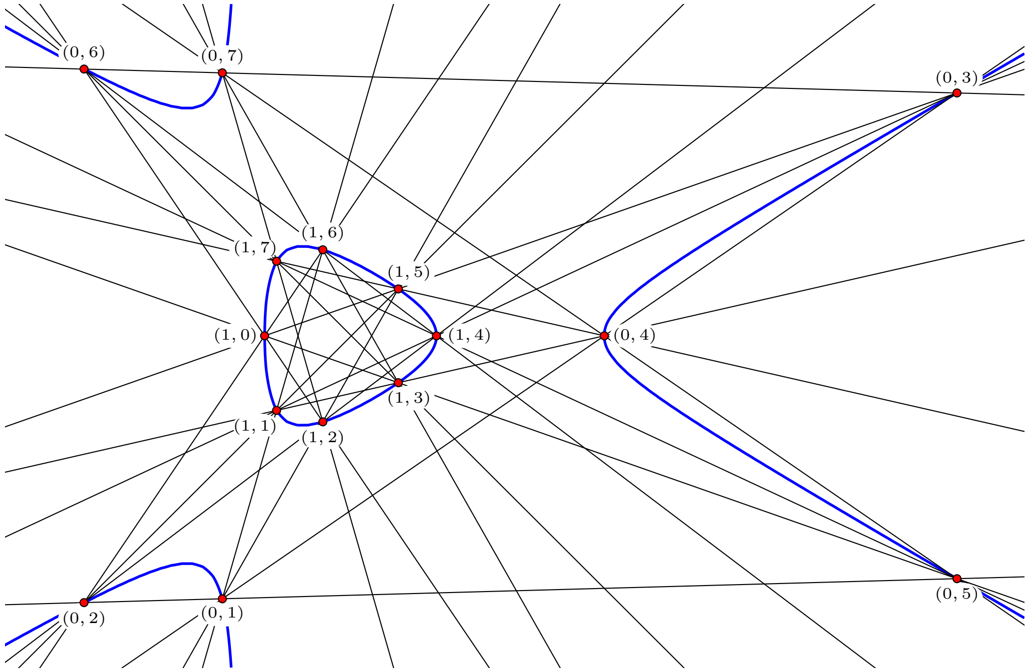


Figure 7: Elliptic D_1-symmetric $(15_4, 20_3)$-configuration derived from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$

Figure 8 shows an elliptic D_3-symmetric $(18_4, 24_3)$-configuration derived from the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$. The group on the elliptic curve has 21 real points and 3 points at infinity. Using only 18 of the real points it is possible to realize a $(18_4, 24_3)$-configuration sitting on two components of the elliptic curve. Recall that we had a $(18_4, 24_3)$-configuration on a one component curve in Figure 4 and another one in Figure 5.

For Figure 9 we started with the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$ with 21 real points and one point at inifinity. Here, an elliptic D_1-symmetric $(21_4, 28_3)$-configuration results. Such a configuration cannot be constructed by the methods presented in Section 3 and Section 4.

Our last example starts with the group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/13\mathbb{Z}$ with 25 real points on the curve and one point at infinity. Omitting the real point corresponding to the group element $(1, 0)$ of order 2, we have 24 real points which carry an elliptic D_1-symmetric $(24_4, 32_3)$-configuration, as shown in Figure 10. Such a configuration cannot be constructed by the methods presented in Section 3 and Section 4.

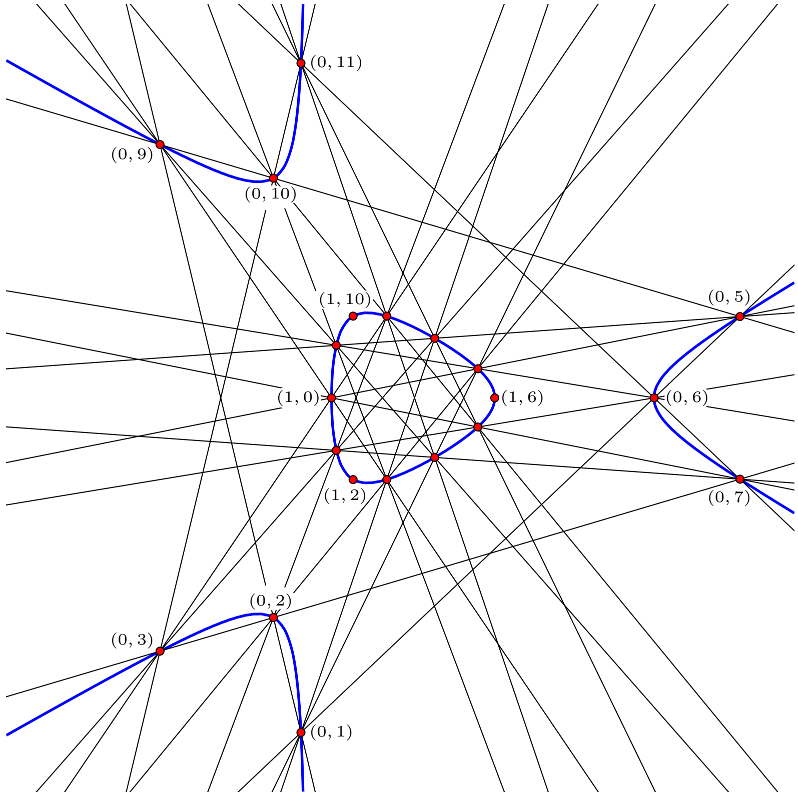


Figure 8: Elliptic D_3-symmetric $(18_4, 24_3)$-configuration derived from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$

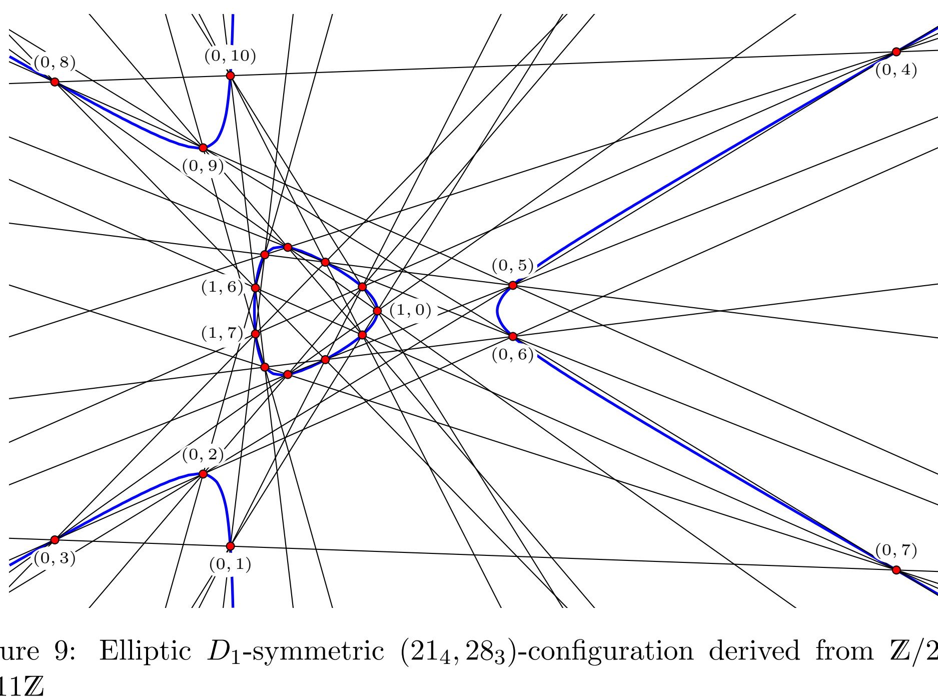


Figure 9: Elliptic D_1-symmetric $(21_4, 28_3)$-configuration derived from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$

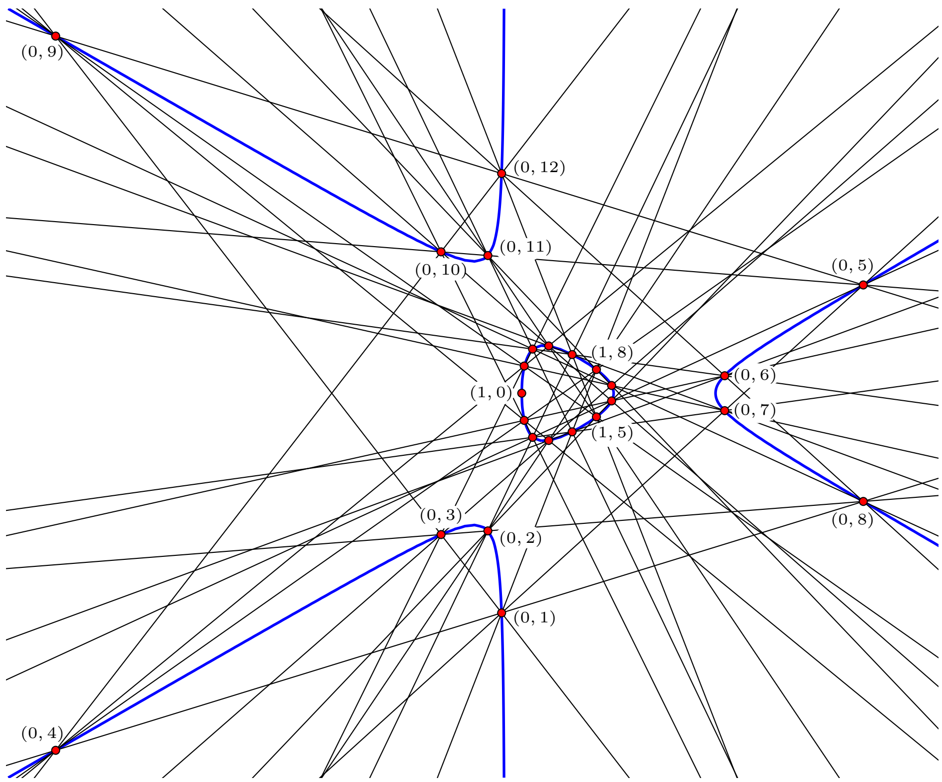


Figure 10: Elliptic D_1-symmetric $(24_4, 32_3)$-configuration derived from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z}$

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