## Chapter 7

## How to Make Two Balls from One

Rests, which are so convenient to the composer and singer, arose for two reasons: necessity and the desire for ornamentation. As for necessity, it would be impossible to sing an entire composition without pausing, for it would cause fatigue that might well prevent a singer from finishing. Rests were adopted also for the sake of ornament. With them parts could enter one after another in fugue or consequence, procedures that give a composition an artful and pleasing effect.

Gioseffo Zarlino
Le Istitutioni Harmoniche, 1558

For two reasons we shall give the reader a rest: one reason is that the reader deserves a pause to reflect on the Axiom of Choice; the other reason is that we would like to show Robinson's beautiful construction-relying on AC—of how to make two balls from one by dividing the ball into only five parts.

## Equidecomposability

Two geometrical figures $A$ and $A^{\prime}$ (i.e., two sets of points lying on the straight line $\mathbb{R}$, on the plane $\mathbb{R}^{2}$, or in the three-dimensional space $\mathbb{R}^{3}$ ) are said to be congruent, denoted $A \cong A^{\prime}$, if $A$ can be obtained from $A^{\prime}$ by translation and/or rotation, but we shall exclude reflections. Two geometrical figures $A$ and $A^{\prime}$ are said to be equidecomposable, denoted $A \simeq A^{\prime}$, if there is a positive integer $n$ and partitions $A=A_{1} \dot{\cup} \ldots \dot{\cup} A_{n}$ and $A^{\prime}=A_{1}^{\prime} \dot{\cup} \ldots \dot{U} A_{n}^{\prime}$ such that for all $1 \leq i \leq n$ : $A_{i} \cong A_{i}^{\prime}$. To indicate that $A$ and $A^{\prime}$ are equidecomposable using at most $n$ pieces we shall write $A \simeq_{n} A^{\prime}$.

Below we shall present two somewhat paradoxical decompositions of the 2-dimensional unit sphere $S_{2}$ as well as of the 3-dimensional solid unit ball $B_{1}$ :

Firstly we show that the unit sphere $S_{2}$ can be partitioned into four parts, say $S_{2}=$ $A \dot{\cup} B \dot{\cup} C \dot{\cup} F$, such that $F$ is countable, $A \cong B \cong C$, and $A \cong B \dot{\cup} C$. This result is known as Hausdorff's Paradox, even though it is just a paradoxical partition of the sphere $S_{2}$ rather than a paradox.

Secondly we show how to make two balls from one, in fact we show that $B_{1} \simeq_{5}$ $B_{1} \cup B_{1}$. This result is due to Robinson and is optimal with respect to the number of pieces needed, i.e., $B_{1} \not 千_{4} B_{1} \cup \dot{\cup} B_{1}$. We would like to mention that about two decades earlier, Banach and Tarski already showed that a unit ball and two unit balls are equidecomposable; however, their construction requires many more than five pieces.
Both decompositions, Hausdorff's partition of the sphere as well as Robinson's decomposition of the ball, rely on the Axiom of Choice. Moreover, it can be shown that in the absence of the Axiom of Choice neither decomposition is provable- but this is beyond the scope of this book (see Related Result 1). However, before we start the constructions, let us briefly discuss the measure-theoretical background of these somewhat paradoxical partitions, in particular of the decomposition of the ball: Firstly, why does Robinson's decomposition of the ball seem paradoxical? Of course, it is because the volume is not preserved; but what are volumes? One could consider the notion of volume as a function $\mu$ which assigns to each set $X \subseteq \mathbb{R}^{3}$ a non-negative real number, called the volume of $A$. We require that the function $\mu$ has the following basic properties:

- $\mu(\emptyset)=0$ and $\mu\left(B_{1}\right)>0$ (e.g., $\mu\left(B_{1}\right)=1$ ),
- $\mu(X \cup Y)=\mu(X)+\mu(Y)$ whenever $X$ and $Y$ are disjoint, and
- $\mu(X)=\mu(Y)$ whenever $X$ and $Y$ are congruent.

Now, by the fact that a unit ball and two unit balls are equidecomposable, and implicitly by Hausdorff's result (see below), we see that there is no such measure on $\mathbb{R}^{3}$, i.e., $\mu$ is not defined for all subsets of $\mathbb{R}^{3}$. Roughly speaking, there are some dust-like subsets of $\mathbb{R}^{3}$ (like the sets we shall construct) to which we cannot assign a volume. Having this in mind, Robinson's decomposition loses its paradoxical character-but certainly not its beauty.

## Hausdorff's Paradox

Before we show how to make two balls from one, we will present Hausdorff's partition of the sphere. The itinerary is as follows: Firstly we define an infinite subgroup $H$ of $\mathrm{SO}(3)$, where $\mathrm{SO}(3)$ is the so-called special orthogonal group consisting of all rotations in $\mathbb{R}^{3}$ leaving the origin fixed. Even though the group $H$ is infinite, it is generated by just two elements. Since $H$ is a subgroup of $\mathrm{SO}(3)$, there is a natural
action of $H$ on the unit sphere $S_{2}$ which induces an equivalence relation on $S_{2}$ by $x \sim y \Longleftrightarrow \exists g \in H(g(x)=y)$ (i.e., $x \sim y$ iff $y$ belongs to the orbit of $x$ ). Then we choose from each equivalence class a representative-this is where the Axiom of Choice comes in-and use the set of representatives to define Hausdorff's partition of the sphere.

We begin the construction by defining the group $H$. For this, consider the following two elements of $\mathrm{SO}(3)$, which will be the generators of $H$ :

$$
\varphi=\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad \psi=\frac{1}{4}\left(\begin{array}{ccc}
-2 & -\sqrt{6} & \sqrt{6} \\
\sqrt{6} & 1 & 3 \\
-\sqrt{6} & 3 & 1
\end{array}\right) .
$$

The linear mapping $\varphi$ is the rotation through $\pi$ about the axis $(0,0,1)$, and $\psi$ is the rotation through $2 \pi / 3$ about the axis $(0,1,1)$. Thus, $\varphi^{2}=\psi^{3}=\iota$ where $\iota$ denotes the identity. We leave it as an exercise to the reader to show by induction on $n$ that for all integers $n \geq 1$ and for all $\varepsilon_{k}= \pm 1$ (where $1 \leq k \leq n$ ) we have

$$
\left(\varphi \psi^{\varepsilon_{n}} \cdots \varphi \psi^{\varepsilon_{1}}\right)=\frac{1}{2^{n+1}}\left(\begin{array}{ccc}
a_{1} & a_{2} \sqrt{6} & a_{3} \sqrt{6} \\
b_{1} \sqrt{6} & b_{2} & b_{3} \\
b_{1}^{\prime} \sqrt{6} & b_{2}^{\prime} & b_{3}^{\prime}
\end{array}\right)
$$

where all numbers $a_{1}, a_{2}, \ldots, b_{3}^{\prime}$ are integers with

- $a_{1} \equiv 2 \bmod 4$,
- $a_{2}, a_{3}, b_{1}, \ldots, b_{3}^{\prime}$ are odd, and
- $b_{1} \equiv b_{1}^{\prime}, b_{2} \equiv b_{2}^{\prime}, b_{3} \equiv b_{3}^{\prime} \bmod 4$.

Hence, we conclude that for all $n \geq 1:\left(\varphi \psi^{\varepsilon_{n}} \cdots \varphi \psi^{\varepsilon_{1}}\right) \notin\{\iota, \varphi\}$. Consequently, for all $n \geq 1$, for all $\varepsilon_{k}= \pm 1$ (where $1 \leq k \leq n$ ), and for $\varepsilon_{0} \in\{0,1\}$ and $\varepsilon_{n+1} \in\{0, \pm 1\}$, we get:

$$
\begin{equation*}
\psi^{\varepsilon_{n+1}} \cdot\left(\varphi \psi^{\varepsilon_{n}} \cdots \varphi \psi^{\varepsilon_{1}}\right) \cdot \varphi^{\varepsilon_{0}} \neq \iota \tag{*}
\end{equation*}
$$

In other words, the only relations between $\varphi$ and $\psi$ are $\varphi^{2}=\psi^{3}=\iota$. Let $H$ be the group of linear transformations-in fact rotations-of $\mathbb{R}^{3}$ generated by the two rotations $\varphi$ and $\psi$. Then $H$ is a subgroup of $\mathrm{SO}(3)$ and every element of $H$ is a rotation which corresponds, by $(*)$, to a unique reduced "word" of the form

$$
\psi^{\varepsilon_{n+1}} \varphi \psi^{\varepsilon_{n}} \cdots \varphi \psi^{\varepsilon_{1}} \varphi^{\varepsilon_{0}}
$$

where $n \geq 0, \varepsilon_{k}= \pm 1$ (for all $1 \leq k \leq n$ ), $\varepsilon_{0} \in\{0,1\}$, and $\varepsilon_{n+1} \in\{0, \pm 1\}$.

We now consider the so-called Cayley graph of $H$ : The Cayley graph of $H$ is a graph with vertex set $H$, where for $\rho_{1}, \rho_{2} \in H$ there is a directed edge from $\rho_{1}$ to $\rho_{2}$ if either $\rho_{2}=\varphi \rho_{1}$ or $\rho_{2}=\psi \rho_{1}$. In the former case, the edge is labelled $\varphi$, in the latter case it is labelled $\psi$, e.g., $\psi \varphi \xrightarrow{\varphi} \varphi \psi \varphi$ or $\psi^{2} \varphi \xrightarrow{\psi} \varphi$.
To each vertex of the Cayley graph of $H$ we assign a label, which is either $\mathbf{1}, \boldsymbol{2}$, or (3. The labelling is done according to the following rules:

- The identity $\iota$ gets the label 1 .
- If $\rho \in H$ is labelled $\boldsymbol{2}$ or (3) and $\sigma=\varphi \rho$, then $\sigma$ is labelled (1).
- If $\rho \in H$ is labelled 1 and $\sigma=\varphi \rho$, then $\sigma$ is labelled either (2) or (3).
- If $\rho \in H$ is labelled (1) (2), or (3) and $\sigma=\psi \rho$, then $\sigma$ is labelled (2) (or (3), or (1), respectively).

These rules are illustrated by the following figures and diagrams:


The lightface label (3) indicates that if $\rho$ is a reduced word in $H$, labelled $\mathbf{1}$, of the form $\psi^{\varepsilon} \rho^{\prime}$ for $\varepsilon= \pm 1$, then $\varphi \rho$ is always labelled (2) (not (3). The following figure shows part of the labelled Cayley graph of $H$ :


The group $H$ acts on the 2-dimensional unit sphere $S_{2}$ and we define the equivalence relation " $\sim$ " on $S_{2}$ via $x \sim y$ iff there is a $\rho \in H$ such that $\rho(x)=y$. The equivalence classes of " $\sim$ " are usually called $H$-orbits, and the $H$-orbit containing $x \in S_{2}$ is written $[x]^{\sim}$. Let $F \subseteq S_{2}$ be the set of all fixed points (i.e., the set of all $y \in S_{2}$ such that there is a $\rho \in H \backslash\{\iota\}$ with $\rho(y)=y$ ). Since $H$ is countable and every rotation $\rho \in H$ has two fixed points, $F$ is countable. We notice first that any point equivalent to a fixed point is a fixed point (i.e., for every $x \in S_{2} \backslash F$ we have $[x]^{\sim} \subseteq S_{2} \backslash F$ ). Indeed, if $\rho(y)=y$ for some $\rho \in H$ and $y \in S_{2}$, then $\sigma \rho \sigma^{-1}(\sigma(y))=\sigma(y)$; that is, if $y$ is fixed for $\rho$, then $\sigma(y)$ is fixed for $\sigma \rho \sigma^{-1}$. Thus, a class of equivalent points consists either entirely of fixed points, or entirely of non-fixed points.
By the Axiom of Choice there is a choice function $f$ for $\mathscr{F}=\left\{[x]^{\sim}: x \in S_{2} \backslash F\right\}$ and let $M=\left\{f\left([x]^{\sim}\right): x \in S_{2} \backslash F\right\}$.
Now we define labels for all non-fixed points (i.e., points in $S_{2} \backslash F$ ) as follows: Firstly, every element in $M$ is labelled (1). Secondly, if $x \in S_{2} \backslash F$, then there is a unique rotation $\rho \in H$ such that $\rho(y)=x$, where $\{y\}=M \cap[x]^{\sim}$. We define the label of the point $x$ by the label of $\rho$ in the labelled Cayley graph of $H$. This induces a partition of $S_{2} \backslash F$ into the following three parts:

$$
\begin{aligned}
& A=\left\{x \in S_{2} \backslash F: x \text { is labelled (1) }\right\}, \\
& B=\left\{x \in S_{2} \backslash F: x \text { is labelled (2) }\right\}, \\
& C=\left\{x \in S_{2} \backslash F: x \text { is labelled (3) } .\right.
\end{aligned}
$$

Thus, $S_{2}=A \dot{\cup} B \dot{\cup} C \dot{\cup} F$ and by the labelling of the vertices of the Cayley graph of $H$ we get

$$
B=\psi[A], \quad C=\psi^{-1}[A], \quad B \dot{\cup} C=\varphi[A] .
$$

Hence, we get $A \cong B, A \cong C$, and $A \cong B \dot{\cup} C$. We leave it as an exercise to the reader to show that this implies $\left(S_{2} \backslash F\right) \simeq_{4}\left(S_{2} \backslash F\right) \dot{\cup}\left(S_{2} \backslash F\right)$.
For each point $x \in S_{2}$ let $l_{x}$ be the line joining the origin (i.e., the centre of the sphere) with $x$, and for $S \subseteq S_{2}$ define $\bar{S}:=\bigcup\left\{l_{x}: x \in S\right\}$. Then the sets $\bar{A}, \bar{B}$, and $\bar{C}$, cannot be Lebesgue measurable (otherwise we would have $0<\mu(\bar{B})=$ $\mu(\bar{C})=\mu(\bar{B} \cup \bar{C})$, a contradiction). In fact, Hausdorff's decomposition shows that there is no non-vanishing measure on $S_{2}$ which is defined for all subsets of $S_{2}$ such that congruent sets have the same measure.

