Chebyshev's theorem on the distribution of prime numbers

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1 The Chebyshev functions

Denote by $\pi(x)$ the number of primes not exceeding $x > 0$. It is well known that there is infinitely many prime numbers, i.e., $\lim_{x\to\infty} \pi(x) \to \infty$. The famous prime number theorem tells us more, namely $\pi(x) \sim x/\log x$.

In this paper, we are going to prove the Chebyshev's theorem, which is an intermediate result of the prime number theory, and use similar methodology to derive a few other interesting results.

Theorem 1 (Euler)**.** *The sum $\sum 1/p$ and the product $\prod(1-1/p)^{-1}$ are both divergent, as p runs through all the prime numbers.*

Proof. We shall first show that the product diverges, and then deduce that the series also does. Let

$$P(x) := \prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-1} \quad \text{and} \quad S(x) := \sum_{p\leqslant x}\frac{1}{p} \quad \text{for } x \in \mathbb{Z}_{\geqslant 2}.$$

Note that for $u \in (0,1)$ and $m \in \mathbb{Z}_{\geqslant 1}$, we have:

$$\frac{1}{1-u} > \frac{1-u^{m+1}}{1-u} = 1 + u + \cdots + u^m > 0.$$

Choose $m \in \mathbb{Z}_{\geqslant 1}$, such that $2^m \geqslant x$. Note that the above inequality holds for all primes $p \leqslant x$ with $u = 1/p$. By multiplying all of the resulting inequalities, we get

$$P(x) > \prod_{p\leqslant x}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^m}\right) \overset{(a)}{\geqslant} \sum_{n=1}^{x}\frac{1}{n} > \int_1^{x+1}\frac{dy}{y} = \log(x+1),$$

where (a) follows from that fact that, if $n = p_1^{m_1}p_2^{m_2}...p_k^{m_k}$ is the prime factorization of $n \in \{1,2,...,x\}$, then $p_1, p_2, ...p_k$ are all $\leqslant x$; and $m_1, m_2, ...m_k$ are all $\leqslant m$ (since $x \geqslant 2^m$). Thus every term on the right hand side is a product of terms on the left hand side.

Hence the product $\prod(1-1/p)^{-1}$ diverges.

To prove the divergence of the series, we consider the expansion:

$$\log\left(\frac{1}{1-u}\right) = \sum_{n=1}^{\infty}\frac{u^n}{n} \quad \text{for } u \in [-1,1).$$

If $u \in (0,1)$, we have:

$$\log\left(\frac{1}{1-u}\right) - u = \sum_{n=2}^{\infty}\frac{u^n}{n} < \sum_{n=2}^{\infty}\frac{u^n}{2} = \frac{u^2}{2(1-u)}.$$

Note that the above equation holds for all primes $p \leqslant x$ with $u = 1/p$.
By adding all of the resulting inequalities, we obtain:

$$\log P(x) - S(x) = \sum_{p \leqslant x} \left(\log \left(\frac{1}{1-1/p} \right) - \frac{1}{p} \right) < \sum_{p \leqslant x} \frac{p^{-2}}{2(1-1/p)} = \frac{1}{2} \sum_{p \leqslant x} \frac{1}{p(p-1)} < \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \frac{1}{2}.$$

It follows that $S(x) > \log P(x) - 1/2 > \log\log x - 1/2$, and hence the sum $\sum 1/p$ also diverges.

□

Definition. The Chebyshev functions ϑ and ψ are defined as follows:

$$\vartheta(x) := \sum_{p \leqslant x} \log p = \sum_{p \in \{q \text{ prime} \,|\, q \leqslant x\}} \log p, \quad x > 0; \tag{1}$$

$$\psi(x) := \sum_{p^m \leqslant x} \log p = \sum_{(p,m) \in \{(q,l) \in \mathbb{N} \times \mathbb{N} \,|\, q \text{ prime}, \, l \geqslant 1 \text{ and } q^l \leqslant x\}} \log p, \quad x > 0. \tag{2}$$

Alternatively, we can define $\psi(x) := \sum_{p \leqslant x} M_p \log p$, where M_p is the highest exponent such that $p^{M_p} \leqslant x$ holds, for example $\psi(10) = 3\log 2 + 2\log 3 + \log 5 + \log 7$.

Note that $M_p = \lfloor \log x / \log p \rfloor$, and hence

$$\psi(x) = \sum_{p \leqslant x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \cdot \log p. \tag{3}$$

Further, it follows from (1) and (2) that $e^{\vartheta(x)}$ equals the product of all primes $p \leqslant x$ and, for $x \geqslant 1, e^{\psi(x)}$ is the least common multiple of all positive integers $n \leqslant x$. If $p^m \leqslant x$, then $p \leqslant x^{1/m}$, and conversely, hence (2) leads to the relation

$$\psi(x) = \vartheta(x) + \vartheta\left(x^{1/2}\right) + \vartheta\left(x^{1/3}\right) + \cdots \tag{4}$$

Recall the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } \exists p \text{ prime}, \, m \in \mathbb{Z}_{\geqslant 1} : n = p^m \\ 0 & \text{otherwise} \end{cases}.$$

From (2) it is immediate that

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n) \tag{5}$$

where the sum is finite because $\forall x < 2: \; \vartheta(x) = 0$.

We shall now establish a connection between the functions

$$\frac{\pi(x)}{x/\log x}, \frac{\vartheta(x)}{x}, \frac{\psi(x)}{x}.$$

Theorem 2. *Let*

$$l_1 = \liminf_{x \to \infty} \frac{\pi(x)}{x/\log x}, \qquad L_1 = \limsup_{x \to \infty} \frac{\pi(x)}{x/\log x},$$

$$l_2 = \liminf_{x \to \infty} \frac{\vartheta(x)}{x}, \qquad L_2 = \limsup_{x \to \infty} \frac{\vartheta(x)}{x},$$

$$l_3 = \liminf_{x \to \infty} \frac{\psi(x)}{x}, \qquad L_3 = \limsup_{x \to \infty} \frac{\psi(x)}{x}.$$

Then $l_1 = l_2 = l_3$, *and* $L_1 = L_2 = L_3$.

Proof. It follows from (3) that, for all $x > 0$,

$$\vartheta(x) = \sum_{p \leqslant x} \log p \leqslant \psi(x) \overset{(3)}{=} \sum_{p \leqslant x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \cdot \log p \leqslant \sum_{p \leqslant x} \frac{\log x}{\log p} \cdot \log p = \sum_{p \leqslant x} 1 \cdot \log x = \pi(x) \log x$$

and hence

$$\frac{\vartheta(x)}{x} \leqslant \frac{\psi(x)}{x} \leqslant \frac{\pi(x) \log x}{x}.$$

For $x \to \infty$, we get

$$\limsup_{x \to \infty} \frac{\vartheta(x)}{x} \leqslant \limsup_{x \to \infty} \frac{\psi(x)}{x} \leqslant \limsup_{x \to \infty} \frac{\pi(x)}{x / \log x}, \quad \text{i.e., } L_2 \leqslant L_3 \leqslant L_1. \tag{6}$$

Now fix $\alpha \in (0,1)$ and $x > 1$. Then

$$\vartheta(x) = \sum_{p \leqslant x} \log p \geqslant \sum_{x^\alpha < p \leqslant x} \log p \geqslant \sum_{x^\alpha < p \leqslant x} \alpha \log x = \alpha \log x \big(\pi(x) - \pi(x^\alpha)\big) > \alpha \log x \big(\pi(x) - x^\alpha\big),$$

Dividing by x on both sides, we get

$$\frac{\vartheta(x)}{x} > \alpha \cdot \frac{\pi(x) \log x}{x} - \alpha x^{\alpha - 1} \log x.$$

Since $\alpha \in (0,1)$, it follows that $x^{\alpha-1} \log x \to 0$, as $x \to \infty$, and thus

$$\limsup_{x \to \infty} \frac{\vartheta(x)}{x} \geqslant \alpha \cdot \limsup_{x \to \infty} \frac{\pi(x)}{x / \log x}, \quad \text{i.e., } L_2 \geqslant \alpha L_1.$$

Because the above inequality holds for all $\alpha \in (0,1)$, we deduce that $L_2 \geqslant L_1$.
Combining this with (6), we get $L_1 = L_2 = L_3$.

The proof of $l_1 = l_2 = l_3$ follows analogously.

□

It follows from Theorem 2 that, if one of the three functions

$$\frac{\pi(x)}{x / \log x}, \quad \frac{\vartheta(x)}{x}, \quad \frac{\psi(x)}{x}$$

tends to a limit as $x \to \infty$, then so do the others, and all three limits are the same. Thus in order to prove the prime number theorem, it is sufficient to show that $\lim_{x \to \infty} \psi(x)/x = 1$.

2 Chebyshev's theorem

Theorem 3 (Chebyshev)**.** *There exist constants a and A, $0 < a < A$, such that*

$$a \cdot \frac{x}{\log x} < \pi(x) < A \cdot \frac{x}{\log x}, \quad \textit{for sufficiently large } x.$$

Proof. Let

$$l = \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x}, \quad L = \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x}$$

We shall prove Theorem 3 by showing that $L \leqslant 4\log 2$, and $l \geqslant \log 2$.

By Theorem 2 these two inequalities are equivalent to

$$L = \limsup_{x\to\infty} \frac{\vartheta(x)}{x} \leqslant 4\log 2; \tag{7}$$

$$l = \liminf_{x\to\infty} \frac{\psi(x)}{x} \geqslant \log 2. \tag{8}$$

Proof of (7). The binomial coefficient

$$N = \binom{2n}{n} = \frac{(n+1)(n+2)\cdots(2n)}{1\cdot 2\cdot 3\cdots n}$$

has the following properties:

$$\text{(i) } N \text{ is an integer, and } N \overset{\text{(a)}}{<} 2^{2n} \overset{\text{(b)}}{<} (2n+1)N \tag{9}$$

where the first inequality follows as N occurs in the binomial expansion of $(1+1)^{2n}$, which has $(2n+1)$ positive terms; while the second from the fact that N is the largest among these $(2n+1)$ terms.

(ii) N is divisible by the product of all primes p, such that $n < p \leqslant 2n$, because every such prime appears in the numerator of N, and not in its denominator.

Because of (ii), we have $N \geqslant \prod_{n<p\leqslant 2n} p$, hence

$$2n\log 2 \overset{(9)}{>} \log N \geqslant \sum_{n<p\leqslant 2n} \log p = \vartheta(2n) - \vartheta(n). \tag{10}$$

If we set $n = 1, 2, 2^2, \cdots, 2^{m-1}$ in (10), and add the resulting inequalities, we get

$$\vartheta(2^m) = \vartheta(2^m) - \vartheta(1) < \sum_{r=1}^{m} 2^r \log 2 < 2^{m+1}\log 2,$$

since $\vartheta(1) = 0$.
Now fix $x > 1$. Choose $m \in \mathbb{Z}$ such that $2^{m-1} \leqslant x < 2^m$. Since ϑ is non-decreasing, (11) gives

$$\vartheta(x) \leqslant \vartheta(2^m) < 2^{m+1}\log 2 \leqslant 4x\log 2, \quad \text{and hence} \quad \frac{\vartheta(x)}{x} < 4\log 2,$$

from which we can deduce

$$L = \limsup_{x\to\infty} \frac{\vartheta(x)}{x} \leqslant 4\log 2,$$

as claimed in (7).

Proof of (8). The second part of theorem is proved differently, for which we need the following lemma.

We say that a prime p divides the integer n exactly k times, if $p^k \mid n$, and $p^{k+1} \nmid n$.

Lemma. *The number of times a prime p exactly divides $m!$ is equal to*

$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \cdots,$$

where the sum above is finite since $\lfloor x \rfloor = 0$ for $0 < x < 1$.

Proof. Among the integers $1, 2, \ldots, m$, there are exactly $\lfloor m/p \rfloor$ which are divisible by p, namely

$$p, 2p, \ldots, \left\lfloor \frac{m}{p} \right\rfloor p.$$

The integers between 1 and m which are divisible by p^2 are

$$p^2, 2p^2, \ldots, \left\lfloor \frac{m}{p^2} \right\rfloor p^2,$$

which are $\lfloor m/p^2 \rfloor$ in number, and so on.

The number of integers between 1 and m which are divisible by p exactly r times, is therefore $\lfloor m/p^r \rfloor - \lfloor m/p^{r+1} \rfloor$. Hence p divides $m!$ exactly

$$\sum_{r \geqslant 1} r \left(\left\lfloor \frac{m}{p^r} \right\rfloor - \left\lfloor \frac{m}{p^{r+1}} \right\rfloor \right) = \sum_{r \geqslant 1} r \left\lfloor \frac{m}{p^r} \right\rfloor - \sum_{r \geqslant 2} (r-1) \left\lfloor \frac{m}{p^r} \right\rfloor = \sum_{r \geqslant 1} \left\lfloor \frac{m}{p^r} \right\rfloor$$

times, which proves the lemma.

□

In order to prove (8), we fix $n \in \mathbb{Z}_{\geqslant 1}$, and consider the integer

$$N = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}.$$

Let p be any prime, such that $p \leqslant 2n$. By virtue of the previous lemma, the numerator of N is divisible by p exactly $\lfloor 2n/p \rfloor + \lfloor 2n/p^2 \rfloor + \cdots$ times, and $n!$ is divisible by p exactly $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots$ times, so that the denominator of N is divisible by p exactly $2\big(\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots\big)$ times.
Hence N is divisible by p exactly v_p times, where

$$v_p = \sum_{r \geqslant 1} \left(\left\lfloor \frac{2n}{p^r} \right\rfloor - 2 \left\lfloor \frac{n}{p^r} \right\rfloor \right)$$

Therefore, since N cannot be divisible by any prime larger than $2n$,

$$N = \prod_{p \leqslant 2n} p^{v_p}.$$

Since $\left\lfloor \frac{2n}{p^r} \right\rfloor = \left\lfloor \frac{n}{p^r} \right\rfloor = 0$ when $p^r > 2n$, i.e., when $r > \lfloor \log 2n / \log p \rfloor$, we have

$$v_p = \sum_{i=1}^{M_p} \left(\left\lfloor \frac{2n}{p^i} \right\rfloor - 2 \left\lfloor \frac{n}{p^i} \right\rfloor \right), \quad \text{where } M_p = \left\lfloor \frac{\log 2n}{\log p} \right\rfloor. \tag{11}$$

Note that, for any $y \in \mathbb{R}$, we have $2\lfloor y \rfloor \leqslant 2y < 2\lfloor y \rfloor + 2$ and $\lfloor 2y \rfloor \leqslant 2y < \lfloor 2y \rfloor + 1$, hence $-1 < \lfloor 2y \rfloor - 2\lfloor y \rfloor < 2$. Since $\lfloor 2y \rfloor - 2\lfloor y \rfloor \in \mathbb{Z}$, we can deduce that

$$\lfloor 2y \rfloor - 2\lfloor y \rfloor \in \{0, 1\}. \tag{12}$$

Combining (11) and (12), we get $v_p \leqslant M_p$, and hence

$$N = \prod_{p \leqslant 2n} p^{v_p} \leqslant \prod_{p \leqslant 2n} p^{M_p}, \quad \text{i.e.,} \quad \log N \leqslant \sum_{p \leqslant 2n} M_p \log p. \tag{13}$$

Recall from (9) that

$$2^{2n} < (2n+1)N, \quad \text{i.e.,} \quad 2n \log 2 < \log(2n+1) + \log N.$$

Combining this result with (3), (11) and (13) yields

$$\psi(2n) \overset{(3)}{=} \sum_{p \leqslant 2n} \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \log p \overset{(11)}{=} \sum_{p \leqslant 2n} M_p \log p \overset{(13)}{\geqslant} \log N > 2n \log 2 - \log(2n+1). \tag{14}$$

Now fix $x > 2$, and set $n = \lfloor x/2 \rfloor \in \mathbb{Z}_{\geqslant 1}$. Then $n > (x/2) - 1$ and $x \geqslant 2\lfloor x/2 \rfloor = 2n$.
Since ψ is non-decreasing, (14) gives

$$\psi(x) \geqslant \psi(2n) \overset{(14)}{>} 2n \log 2 - \log(2n+1) > (x-2)\log 2 - \log(x+1).$$

Dividing by x on both sides, we get

$$\frac{\psi(x)}{x} > \frac{x-2}{x} \cdot \log 2 - \frac{\log(x+1)}{x}.$$

Since $(x-2)/x \to 1$ and $x^{-1}\log(x+1) \to 0$, as $x \to \infty$, we obtain the desired inequality:

$$l = \liminf_{x \to \infty} \frac{\psi(x)}{x} \geqslant \log 2.$$

□

Corollary. *Let p_n be the n-th prime. There exist constants c and C, $0 < c < C$, such that:*

$$c \cdot \log\log x < \sum_{n=1}^{x} \frac{1}{p_n} < C \cdot \log\log x, \quad \textit{for sufficiently large } x.$$

Proof. Theorem 3 tells us that there exist constants a and A, $0 < a < A$, such that

$$a \cdot \frac{x}{\log x} < \pi(x) < A \cdot \frac{x}{\log x}, \quad \text{for sufficiently large } x.$$

Let $n_0 \in \mathbb{Z}_{\geqslant 3}$ be large enough, such that $\forall n \geqslant n_0$: (i) $a \cdot p_n/\log p_n < \pi(p_n) < A \cdot p_n/\log p_n$,
(ii) $\log p_n < a\sqrt{p_n}$.

Fix $n \geqslant n_0$. Note that $n < p_n$ and $\pi(p_n) = n$, hence

$$n \log n < \pi(p_n) \log p_n \overset{\text{(i)}}{<} A \cdot p_n, \tag{15}$$

$$\sqrt{p_n} \overset{\text{(ii)}}{<} a \cdot \frac{p_n}{\log p_n} \overset{\text{(i)}}{<} \pi(p_n) = n. \tag{16}$$

Thus $\log p_n < 2 \log n$, and hence

$$a p_n \overset{(16)}{<} n \log p_n < 2n \log n, \tag{17}$$

$$\frac{a}{2n \log n} \overset{(17)}{<} \frac{1}{p_n} \overset{(15)}{<} \frac{A}{n \log n}. \tag{18}$$

From (18) we can deduce the corollary, because of the following calculation:

$$
\begin{aligned}
\forall x \geqslant e^{n_0}: \quad \sum_{n=1}^{x} \frac{1}{p_n} &= \sum_{n=n_0+1}^{\lfloor x \rfloor} \frac{1}{p_n} + \sum_{n=1}^{n_0} \frac{1}{p_n} \\
&\overset{(18)}{<} A \cdot \sum_{n=n_0+1}^{\lfloor x \rfloor} \frac{1}{n \log n} + \sum_{n=1}^{p_n} \frac{1}{n} \\
&\overset{(16)}{<} A \cdot \sum_{n=n_0+1}^{\lfloor x \rfloor} \frac{1}{n \log n} + \sum_{n=1}^{{n_0}^2} \frac{1}{n} \\
&\leqslant A \cdot \int_{n_0}^{\lfloor x \rfloor} \frac{dy}{y \log y} + 1 + \int_{1}^{{n_0}^2} \frac{dy}{y} \\
&= A \cdot \log\log\lfloor x \rfloor - A \cdot \log\log n_0 + 1 + \log {n_0}^2 \\
&\leqslant A \cdot \log\log x + 1 + 2\log n_0 \\
&\overset{(b)}{\leqslant} A \cdot \log\log x + \log\log x + 2\log\log x \\
&= (A+3) \cdot \log\log x,
\end{aligned}
$$

where (b) follows from $x \geqslant e^{n_0}$, i.e., $\log\log x \geqslant \log\log e^{n_0} = \log n_0$;

$$
\begin{aligned}
\forall x \geqslant {n_0}^{\log n_0}: \quad \sum_{n=1}^{x} \frac{1}{p_n} &\geqslant \sum_{n=n_0}^{x} \frac{1}{p_n} \\
&\overset{(18)}{>} \frac{a}{2} \cdot \sum_{n=n_0}^{x} \frac{1}{n \log n} \\
&\geqslant \frac{a}{2} \cdot \int_{n_0}^{\lfloor x \rfloor + 1} \frac{dy}{y \log y} \\
&= \frac{a}{2} \cdot \log\log(\lfloor x \rfloor + 1) - \frac{a}{2} \cdot \log\log n_0 \\
&\overset{(c)}{\geqslant} \frac{a}{2} \cdot \log\log x - \frac{a}{4} \cdot \log\log x \\
&= \frac{a}{4} \cdot \log\log x,
\end{aligned}
$$

where (c) follows from $x \geqslant {n_0}^{\log n_0}$, i.e., $\log\log x > \log\log {n_0}^{\log n_0} = \log(\log n_0 \cdot \log n_0) = 2\log\log n_0$.

□

3 Bertrand's postulate

Theorem 4. *Let $n \in \mathbb{N}$. Then there is some prime p such that $n < p \leq 2n$.*

The first proof is due to Chebyschev, the proof presented here is due to S.S.Pillai.

Instead of the estimate $\frac{2^{2n}}{2n+1} < N < 2^{2n}$ for $N = \binom{2n}{n}$ employed in the proof of Chebyschev's theorem we will use instead the sharper estimate

$$\frac{2^{2n}}{2\sqrt{n}} < N < \frac{2^{2n}}{\sqrt{2n}}(n \geq 2), \tag{19}$$

in order to show that

$$\vartheta(n) < 2n \log 2 (n \geq 1). \tag{20}$$

We will first prove the inequalities in (19).

Define

$$P = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \cdot \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} = \frac{(2n)!}{2^{2n}(n!)^2},$$

and so $N = 2^{2n}P$

Now

$$1 > \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{(2n)^2}\right)$$

$$\implies 1 > \left(\frac{1 \cdot 3}{2^2}\right)\left(\frac{3 \cdot 5}{4^2}\right)\left(\frac{5 \cdot 7}{6^2}\right)\ldots\left(\frac{(2n-1)(2n+1)}{(2n)^2}\right)$$

$$\implies 1 > (2n+1)P^2 > 2nP^2 = \frac{2n}{2^{4n}}N^2.$$

From this we directly obtain the second inequality in (19).

For the first one we have

$$1 > \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\ldots\left(1 - \frac{1}{(2n-1)^2}\right)$$

$$\implies 1 > \left(\frac{2 \cdot 4}{3^2}\right)\left(\frac{4 \cdot 6}{5^2}\right)\ldots\left(\frac{(2n-2)2n}{(2n-1)^2}\right)$$

$$\implies 1 > \frac{1}{4nP^2} = \frac{2^{4n}}{4nN^2}.$$

This then gives us the first inequality, and (19) is proved.

We then prove (20). This trivially holds for $n = 1, 2$. Now we assume that it holds for some $n \geq 2$, we shall show that $\vartheta(2n-1) < 2(2n-1)\log 2$, which would imply $\vartheta(2n) = \vartheta(2n-1) < 4n \log 2$.

Consider the integer

$$\frac{N}{2} = \frac{1}{2}\binom{2n}{n} = \frac{(2n)!}{(n!)^2}\frac{n}{2n} = \frac{(2n-1)!}{n!(n-1)!} = \binom{2n-1}{n-1}.$$

This is divisible by every prime p with $n < p \leq 2n-1$, and so also by their product, which gives:

$$\frac{N}{2} \geq \prod_{n<p\leq 2n-1} p \implies \log\frac{N}{2} \geq \vartheta(2n-1) - \vartheta(n).$$

(19) gives us on the other hand

$$\log N < 2n \log 2 - \frac{1}{2}\log 2n.$$

Hence, by combining the two, we have

$$\vartheta(2n-1) - \vartheta(n) < (2n-1)\log 2 - \frac{1}{2}\log 2n.$$

By our assumption $\vartheta(n) < 2n\log 2$ and so

$$\vartheta(2n-1) < 2n\log 2 + (2n-1)\log 2 - \frac{1}{2}\log 2n.$$

This implies when $n \geq 2$

$$\vartheta(2n-1) < 2(2n-1)\log 2,$$

which is exactly the sought inequality. Hence, if (20) holds for n, it holds for $2n-1$ and by the previous remark also for $2n$. This means that, if it holds for every $n \in (2^{r-1}, 2^r], r \geq 1$, then it holds for every $n \in (2^r, 2^{r+1}]$, and so the claim follows by the induction since it holds for 1 and 2.

Now using these results we shall prove the theorem in two parts, one part for the case $n \geq 2^6$ and one for $n < 2^6$. Starting with the first,recall from a previous proof that we have

$$N = \binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \prod_{p \leq 2n} p^{v_p}$$

with

$$v_p = \sum_{r \geq 1}\left(\left\lfloor \frac{2n}{p^r}\right\rfloor - 2\left\lfloor \frac{n}{p^r}\right\rfloor\right).$$

Then

$$\log N = \sum_{p \leq 2n} v_p \log p.$$

We partition this sum into the following value ranges for p, calling them $\sum_1, \sum_2, \sum_3 \sum_4$ respectively:

1. $n < p \leq 2n$
2. $\frac{2n}{3} < p \leq n$
3. $\sqrt{2n} < p \leq \frac{2n}{3}$ with $n \geq 5$
4. $p \leq \sqrt{2n}$.

For $\sum_1$ we have $\frac{n}{p} < 1$, so $\lfloor \frac{n}{p} \rfloor = 0$ and $1 \leq \frac{2n}{p} < 2$ so that $\lfloor \frac{2n}{p} \rfloor = 1$ and $\lfloor \frac{2n}{p^2} \rfloor = 0$. Then $v_p = 1$ and we get

$$\sum_1 = \sum_{n<p\leq 2n} v_p \log p = \sum_{n<p\leq 2n} \log p = \vartheta(2n) - \vartheta(n).$$

For $\sum_2$ we have that $1 \leq \frac{n}{p} \leq \frac{3}{2}$, so $\lfloor \frac{n}{p} \rfloor = 1$ and $\lfloor \frac{2n}{p} \rfloor = 2$; for $n \geq 3$ we get $\lfloor \frac{2n}{p^2} \rfloor = 0$, giving us in total $\sum_2 = 0$ for $n \geq 3$.

For $\sum_3$ we have $n \geq 5$ and $\frac{n}{p^2} < \frac{2n}{p^2} < 1$, so $v_p = \lfloor \frac{2n}{p} \rfloor - 2\lfloor \frac{n}{p} \rfloor = 0$ or 1. Therefore

$$\sum_3 \leq \sum_{\sqrt{2n}<p\leq 2n/3} \log p = \vartheta\left(\frac{2n}{3}\right) - \vartheta(\sqrt{2n}).$$

Now

$$\vartheta(\sqrt{2n}) = \sum_{p\leq\sqrt{2n}} \log p \geq \log 2 \sum_{p\leq\sqrt{2n}} 1 = \pi(\sqrt{2n})\log 2,$$

and so

$$\sum_3 \leq \vartheta\left(\frac{2n}{3}\right) - \pi(\sqrt{2n})\log 2.$$

Finally for $\sum_4$ we use (13) and obtain

$$v_p \leq M_p = \left\lfloor \frac{\log 2n}{\log p} \right\rfloor,$$

so that

$$\sum_4 \leq \sum_{p \leq 2n} M_p \log p \leq \sum_{p \leq \sqrt{2n}} \frac{\log 2n}{\log p} \cdot \log p = \log 2n \sum_{p \leq \sqrt{2n}} 1.$$

Hence

$$\sum_4 \leq \pi(\sqrt{2n}) \log 2n.$$

We add up all the sums together and get, for $n \geq 5$,

$$\log N \leq \vartheta(2n) - \vartheta(n) + \vartheta\left(\frac{2n}{3}\right) - \pi(\sqrt{2n})(\log 2 - \log 2n). \tag{21}$$

We will use this to show that $\vartheta(2n) - \vartheta(n) > 0$ for large enough n. For this purpose we use the following three inequalities:

1. $\log N > 2n \log 2 - \log(2\sqrt{n})$ which follows from first inequality of (19);
2. $\vartheta\left(\frac{2n}{3}\right) = \vartheta\left(\left\lfloor \frac{2n}{3} \right\rfloor\right) < 2\left\lfloor \frac{2n}{3} \right\rfloor \log 2$ for $n \geq 2$ which follows from (20);
3. $\pi(n) \leq \frac{n}{2}$ for $n \geq 8$, since all even numbers greater than 2 are composite.

Combining these three inequalities with (21) we obtain, for $n \geq 32$,

$$\vartheta(2n) - \vartheta(n) > 2n \log 2 - \log(2\sqrt{n}) - \frac{4n}{3} \log 2 - \frac{\sqrt{2n}}{2} \log n$$

$$\implies \vartheta(2n) - \vartheta(n) > \left(\frac{2n}{3} - 1\right) \log 2 - \left(\frac{\sqrt{2n}+1}{2}\right) \log n.$$

It remains to show that this is strictly positive.

This can be manually checked and verified for $n = 2^6$, now we check it for $n > 2^6$. For this purpose we will rewrite the inequality as

$$\sqrt{2n} - \frac{3}{2}\frac{\log n}{\log 2} - \frac{3\sqrt{2}}{\log 2}\frac{\log \sqrt{4n}}{\sqrt{4n}} > 0.$$

Treating this as two real functions of the form

$$\sqrt{2x} - \frac{3}{2}\frac{\log x}{\log 2} \quad \text{and} \quad -\frac{3\sqrt{2}}{\log 2}\frac{\log \sqrt{4x}}{\sqrt{4x}},$$

notice that both functions have positive derivatives for $x \geq 2^6$ and their sum is positive for $x = 2^6$, hence the sum is positive for all $x > 2^6$ and so in particular for every natural number in that range.

Then it follows that $\vartheta(2n) - \vartheta(n) > 0$ for $n \geq 2^6$,and so Bertrand's postulate follows for $n \geq 64$.

Now for the smaller naturals, consider the primes $2, 3, 5, 7, 13, 23, 43, 67$. Each one of these is smaller than twice the previous one, in particular for any $n < 64$ we may pick one of these primes, which will satisfy Bertrand's postulate.

Thus we have proven that the postulate holds for every natural number n.

4 Asymptotic bounds for $\frac{\pi(n)}{n/\log n}$

In this section we will prove the following theorem:

Theorem 5.

$$\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \leq 1 \leq \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x}.$$

Theorem 5 implies that, if the limit $\lim_{n\to\infty} \frac{\pi(n)}{n/\log n}$ exists, then it is equal to 1.

We will prove instead that

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \leq 1 \leq \limsup_{x\to\infty} \frac{\psi(x)}{x},$$

which by Theorem 2 is equivalent to the assertion in Theorem 5.

Let $f(s) = -\frac{\zeta'(s)}{\zeta(s)}$ for every real $s > 1$, and let

$$l = \liminf_{x\to\infty} \frac{\psi(x)}{x} \ , \ L = \limsup_{x\to\infty} \frac{\psi(x)}{x}$$

$$l' = \liminf_{s\to 1^+} (s-1)f(s) \ , \ L' = \limsup_{s\to 1^+} (s-1)f(s)$$

Obviously $l \leq L$ and $l' \leq L'$.

We will show that $l \leq l' \leq L' \leq L$ and $l' = L' = 1$; together these two give us Theorem 5.

Choose some arbitrary real $B > L$, then $\frac{\psi(x)}{x} < B$ for all $x \geq x_0 = x_0(B)$, and we can assume without loss of generality $x_0 > 1$.

Here we will need a statement of Abel's Theorem, whose proof we will omit:

Theorem 6. *Let $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ be a sequence of real numbers, such that $\lambda_n \to \infty$ as $n \to \infty$ and let (a_n) be a sequence of complex numbers. Let $A(x) = \sum_{\lambda_n \leq x} a_n$, and $\phi(x)$ a complex-valued function defined for $x \geq 0$. Then*

$$\sum_{n=1}^{k} a_n \phi(\lambda_n) = A(\lambda_k)\phi(\lambda_k) - \sum_{n=1}^{k-1} A(\lambda_n)(\phi(\lambda_{n+1}) - \phi(\lambda_n)). \tag{22}$$

If ϕ has a continuous derivative in $(0,\infty)$, and $x \geq \lambda_1$, then equation 22 can be written as

$$\sum_{\lambda_n \leq x} a_n \phi(\lambda_n) = A(x)\phi(x) - \int_{\lambda_1}^{x} A(t)\phi'(t)dt. \tag{23}$$

If, in addition, $A(x)\phi(x) \to 0$ as $x \to \infty$, then

$$\sum_{n=1}^{\infty} a_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} A(t)\phi'(t)dt, \tag{24}$$

provided either side is convergent.

Now for $s > 1$, set $\lambda_n = n$, $a_n = \Lambda(n)$, and $\phi(x) = x^{-s}$, then $A(x) = \psi(x)$, and $A(x)\phi(x) \to \infty$ as $x \to \infty$, since $\psi(x) \leq \pi(x)\log x \leq x\log x$, so that $A(x)\phi(x) = O(x^{1-s}\log x) = o(1)$. We will also use without proof that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \ (s \text{ real}, \ s > 1).$$

Combining this with equation 24 gives

$$f(s) = s\int_1^{\infty} \frac{\psi(x)}{x^{s+1}}dx < s\int_1^{x_0} \frac{\psi(x)}{x^{s+1}}dx + \int_{x_0}^{\infty} \frac{B}{x^s}dx,$$

so that

$$f(s) < s\int_1^{x_0} \frac{\psi(x)}{x^{s+1}}dx + s\int_1^{\infty} \frac{B}{x^s}dx < s\int_1^{x_0} \frac{\psi(x)}{x^2}dx + \frac{sB}{s-1}.$$

We set

$$\int_1^{x_0} \frac{\psi(x)}{x^2}dx = K = K(x_0) = K(x_0, B),$$

so that we can rewrite the above equation in the form

$$(s-1)f(s) < s(s-1)K + sB.$$

We let $s \to 1^+$, obtaining

$$L' = \limsup_{s\to 1+0}(s-1)f(s) \le B,$$

and since $B > L$ was arbitrary we have necessarily $L' \le L$. An analogous argument gives $l \le l'$, and so $l \le l' \le L' \le L$

Now we prove $l' = L' = 1$ by showing

$$\lim_{s\to 1^+} -(s-1)^2\zeta'(s) = 1 \text{ and } \lim_{s\to 1^+}(s-1)\zeta(s) = 1.$$

This implies $(s-1)f(s) \to 1$ as $s \to 1^+$.

For $s > 1$, we have that x^{-s} is decreasing as a function of x, and so

$$\int_1^{\infty} \frac{dx}{x^s} < \sum_{n=1}^{\infty} \frac{1}{n^s} < 1 + \int_1^{\infty} \frac{dx}{x^s};$$

hence

$$\frac{1}{s-1} < \zeta(s) < \frac{s}{s-1},$$

implying $(s-1)\zeta(s) \to 1$ as $s \to 1^+$.

Now for the second part we have that, for $s > 1$ and $x \ge e$, the function $x^{-s}\log x$ is decreasing, so

$$-\zeta'(s) = \sum_{n=1}^{\infty} \frac{\log n}{n^s} = \int_1^{\infty} \frac{\log x}{x^s}dx + O(1).$$

By substituting $x^{s-1} = e^y$ we get

$$-\zeta'(s) = \frac{1}{(s-1)^2}\int_0^{\infty} ye^{-y}dy + O(1) = \frac{1}{(s-1)^2} + O(1),$$

so that

$$(s-1)f(s) = -\frac{(s-1)^2\zeta'(s)}{(s-1)\zeta(s)} \to 1 \text{ as } s \to 1^+.$$

.

This means that $l' = L' = 1$ and thus $l \le 1 \le L$, concluding the proof of the Theorem.