Weyl's theorems on uniform distribution and Kronecker's theorem

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1 Introduction

In the third presentation of this seminar we have seen a classical theorem of Dirichlet about the approximation of irrational numbers by rational numbers:

Theorem 1.1. *For every irrational number $\xi \in \mathbb{R} \setminus \mathbb{Q}$ there exist infinitely many rational numbers $\frac{p}{q} \in \mathbb{Q}$, such that*

$$|\xi - \frac{p}{q}| < \frac{1}{q^2}.$$

From this theorem follows the following theorem of Dirichlet:

Theorem 1.2 (Dirichlet's theorem)**.** *For every $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and for every $\epsilon > 0$, there exist infinitely many integers $p, q \in \mathbb{Z}$ such that*

$$|q\xi - p| < \epsilon.$$

Proof. Let $0 < \epsilon < 1$ and consider the integer $1 + [\frac{1}{\epsilon}]$. Since we have infinitely many rationals $\frac{p}{q}$ such that $|q\xi - p| < \frac{1}{q}$,by Theorem 1.1, it follows that there exist infinitely many fractions $\frac{p}{q}$ with denominator $q \geq 1 + [\frac{1}{\epsilon}]$ for which we have

$$|q\xi - p| < \frac{1}{q} \leq \frac{1}{1 + [\frac{1}{\epsilon}]} \leq \frac{1}{1 + (\frac{1}{\epsilon} - 1)} = \epsilon.$$

□

We can generalize this result to obtain:

Theorem 1.3. *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}, \alpha \in \mathbb{R}, N, \epsilon \in \mathbb{R}_{>0}$ be given. Then there exists numbers $n, p \in \mathbb{Z}$ such that $n > N$ and $|n\theta - p - \alpha| < \epsilon$.*

This generalization of Dirichlet's theorem is itself a special case of a deeper result due to Hermann Weyl on the *uniform distribution* of numbers, which we will see and prove in this chapter. Let us state a few remarks about this statement:

- For $\alpha = 0$, the assertion in the theorem reduces to Dirichlet's theorem.
- For $0 < \alpha < 1$ and $\epsilon > 0$ arbitrary small, it follows that the fractional part of $n\theta$, namely $\{n\theta\} := n\theta - [n\theta]$, is arbitrary close to α. In other words, the set of numbers $\{n\theta\}, n = 1, 2, ...$ is everywhere dense in $[0, 1)$. We will later see the proof of this statement as a corollary to Theorem 4.3.

Since we are going to deal with the fractional part of real numbers, we introduce the following definition for convenience:

Definition. *The real numbers x_1, x_2 are said to be* ***congruent modulo 1*** *if $x_1 - x_2 \in \mathbb{Z}$.*

This clearly defines an equivalence relation, which partitions all real numbers in equivalence classes consisting of all real numbers with the same fractional part. The map $x \to e^{2\pi i x}$ clearly induces a one-to-one correspondence between the set of equivalence classes and the unit circle $\{z \in \mathbb{C} | \, |z| = 1\}$.

2 Uniform distribution in the unit interval

We start this section with a couple of necessary definitions. Let S be a finite set of real numbers $\alpha_1, ..., \alpha_Q$ contained in $[0,1)$, that is $0 \leq \alpha_j < 1, 1 \leq j \leq Q$.

Definition. *We define an* ***interval function*** $\phi(a,b)$ *for any* $0 \leq a < b \leq 1$*, via:*

$$\phi(a,b) := |\{\alpha_j : a \leq \alpha_j < b, 1 \leq j \leq Q\}| \tag{1}$$

Hence $\phi(a,b)$ *defines the number of* $\alpha_j's$ *contained in* $[a,b)$.

Definition. *Furthermore we define the* ***discrepancy of S*** *via:*

$$D := \sup_{a,b} \left| \frac{\phi(a,b)}{Q} - (b-a) \right|. \tag{2}$$

Since $\dfrac{\phi(a,b)}{Q}$ takes values in $[0,1]$ and $b-a$ take values in $(0,1]$, we clearly have that $D \leq 1$. In addition we can always choose a and b so that $0 < D$, for instance if we have a and b so that $\dfrac{\phi(a,b)}{Q} = b - a$, then we can take a slightly bigger interval $[a', b')$ containing $[a,b)$, which includes as much points of the sequence as $[a,b)$ and thus $\dfrac{\phi(a,b)}{Q} = \dfrac{\phi(a',b')}{Q} < b' - a'$, which implies that $0 < D$. Therefore we have that $0 < D \leq 1$. Now let $I := [a,b)$, $|I| = (b-a), \phi(I) := \phi(a,b)$. Then we can write D as

$$D = \sup_{a,b} \left| \frac{\phi(I)}{Q} - |I| \right|. \tag{3}$$

Let now $(\alpha_i)_{i \in \mathbb{N}}$ be an infinite sequence of real numbers in $[0,1)$ and define D_n to be the discrepancy of the first n terms of $(\alpha_i)_{i \in \mathbb{N}}$.

Definition. *We say that the sequence* $(\alpha_i)_{i \in \mathbb{N}}$ *is uniformly distributed if* $D_n \to 0$ *as* $n \to \infty$.

Let $\phi_n(a,b) = |\{\alpha_j : a \leq \alpha_j < b, 1 \leq j \leq n\}|$.

Lemma. *If* $(\alpha_i)_{i \in \mathbb{N}}$ *is uniformly distributed in* $[0,1)$*, we get that:*

$$\frac{\phi_n(a,b)}{n} \to (b-a) \tag{4}$$

as $n \to \infty$ $\forall a, b \in \mathbb{R}$ *with* $0 \leq a < b \leq 1$.

Proof. Let $(\alpha_i)_{i \in \mathbb{N}}$ be uniformly distributed in $[0,1)$. By definition we get that

$$\lim_{n \to \infty} D_n = \lim_{n \to \infty} \sup_{a,b} \left| \frac{\phi_n(a,b)}{n} - (b-a) \right| = 0,$$

which implies that

$$\frac{\phi_n(a,b)}{n} \to (b-a)$$

as $n \to \infty$ $\forall a, b \in \mathbb{R}$ with $0 \le a < b \le 1$.

□

The converse also holds: If (4) holds for every interval $[a, b)$, then $(\alpha_i)_{i\in\mathbb{N}}$ is uniformly distributed. We sum up by stating the following theorem:

Theorem 2.1. *Let $(\alpha_i)_{i\in\mathbb{N}}$ be an infinite sequence in $[0,1)$. $(\alpha_i)_{i\in\mathbb{N}}$ is uniformly distributed if and only if*

$$\frac{\phi_n(a,b)}{n} \to (b-a)$$

as $n \to \infty$ for every $0 \le a < b \le 1$.

Proof. (⇒) This direction follows directly from the definition of uniformly distributed sequences, as already mentioned.

(⇐) Suppose now that (4) holds. Fix $\delta \in (0,1)$; we can split $[0,1)$ into finitely many subintervals (I_k) of length δ. Now given any interval $[c,d) \subset [0,1)$, denote by r the number of subintervals I_k which lie in the interior of $[c,d)$. Their total length is then $r\delta$, and we have that

$$r\delta > (d-c) - 2\delta,$$

because if $r\delta \le (d-c) - 2\delta$, then there must be room for at least one more interval I_k which fits into $[c,d) \setminus \bigcup_k I_k$. Let r' be the number of intervals intersecting $[c,d)$, then

$$r'\delta < (d-c) + 2\delta,$$

because if $r'\delta \ge (d-c) + 2\delta$, then there exists at least one interval I_k which doesn't intersect $[c,d)$. Since (4) holds for each interval $[a,b)$, it holds in particular for any interval I_k of length δ. Since there are finitely many intervals I_k, for every $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that for every k,

$$\delta - \epsilon \le \frac{\phi_n(I_k)}{n} \le \delta + \epsilon, \forall n > N(\epsilon).$$

If we now choose $\epsilon = \delta^2$, we get

$$(1-\delta)\delta \le \frac{\phi_n(I_k)}{n} \le (1+\delta)\delta, \forall n > N'(\delta).$$

Thus we have the following chain of inequalities:

$$r\delta(1-\delta) \le \frac{\phi_n(I_1) + ... + \phi_n(I_r)}{n} \le \frac{\phi_n(c,d)}{n} \le \frac{\phi_n(I_1) + ... + \phi_n(I_{r'})}{n} \le r'\delta(1+\delta), \forall n > N'(\delta).$$

Now, using $(d-c)-2\delta < r\delta$ and $(d-c)+2\delta > r'\delta$, we obtain:

$$((d-c)-2\delta)(1-\delta) \leq \frac{\phi_n(c,d)}{n} \leq ((d-c)+2\delta)(1+\delta). \tag{5}$$

Let us look at these two inequalities separately. By using the fact that $d-c \leq 1$, we get from equation (5):

$$-2\delta(1-\delta)-\delta(d-c) \leq \frac{\phi_n(c,d)}{n}-(d-c) \leq 2\delta(1+\delta)+\delta(d-c)$$

$$\Rightarrow \left|\frac{\phi_n(c,d)}{n}-(d-c)\right| \leq 2\delta(1+\delta)+\delta(d-c) < 2\delta(1+\delta)+\delta = 3\delta+2\delta^2.$$

Therefore, for every interval $[c,d) \subset [0,1)$ and for every $0<\delta<1$, it holds that

$$\left|\frac{\phi_n(c,d)}{n}-(d-c)\right| \leq 3\delta+2\delta^2 \forall n > N'(\delta).$$

Taking the supremum over all intervals, this implies that $D_n \to 0$ as $n \to \infty$, showing the converse implication. □

At this point we remark that a uniformly distributed sequence $(\alpha_j)_{j\in\mathbb{N}}$ is dense in $[0,1)$.

Proof. Suppose by contradiction that $(\alpha_j)_{j\in\mathbb{N}}$ is not dense in $[0,1)$. Hence we can find an non-empty open subinterval $(c,d) \subset [0,1)$ such that $(c,d) \cap (\alpha_j)_{j\in\mathbb{N}} = \emptyset$. Therefore for and $\epsilon > 0$ small enough, we have that $(c-\epsilon,d) \cap (\alpha_j)_{j\in\mathbb{N}} = \emptyset$. Using the fact that $(\alpha_j)_{j\in\mathbb{N}}$ is uniformly distributed, we obtain

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} \sup_{a,b} \left|\frac{\phi_n(a,b)}{n}-(b-a)\right| = 0,$$

which yields a contradiction, since

$$\lim_{n\to\infty} \left|\frac{\phi_n(c-\epsilon,d)}{n}-(d-(c-\epsilon))\right| = d-(c-\epsilon) > 0,$$

where we used that $\phi_n(c-\epsilon,d)=0$ for all $n \in \mathbb{N}$. □

3 Uniform distribution modulo 1

Definition. *An infinite sequence of real numbers $(\alpha_j)_{j\in\mathbb{N}}$ (not necessarily contained in $[0,1)$) is said to be **uniformly distributed modulo 1**, if the corresponding sequence of fractional parts $(\{\alpha_j\})_{j\in\mathbb{N}}$ is uniformly distributed.*

Thus, if D_n denotes the discrepancy of the first n terms of $(\{\alpha_j\})_{j\in\mathbb{N}}$, then $D_n \to 0$ as $n \to \infty$.
We will see that the condition of D_n going to 0 as $n \to \infty$ has an alternative but equivalent formulation in terms of *discrepancy modulo 1.*

Let $S = \{\alpha_1, ..., \alpha_Q\}$ be a set of real numbers, and let T denote the set of real numbers of the form $\alpha_k + t$, $1 \leq k \leq Q$, $t \in \mathbb{Z}$. Given $a, b \in \mathbb{R}$, $b \geq a$, define $\phi^*(a,b) := |\{t \in T : t \in [a,b)\}|$.
Then $\phi^*(a+t, b+t) = \phi^*(a,b)$ for every $t \in \mathbb{Z}$ and further $\phi^*(a,b) = \phi(a,b)$ if $0 \leq a < b \leq 1$, where $\phi(a,b)$ for $(\{\alpha_j\})_{1\leq j\leq Q}$, is defined as before.

Definition. *The* ***discrepancy mod 1*** *of a set* $S = \{\alpha_1, ..., \alpha_Q\}$ *of real numbers is defined to be*

$$D^* := \sup_{0\leq b-a\leq 1} \left| \frac{\phi^*(a,b)}{Q} - (b-a) \right|,$$

where a runs through all real numbers. Due to the properties of ϕ^**, namely* $\phi^*(a+t, b+t) = \phi^*(a,b)$ *for every* $t \in \mathbb{Z}$*, we may equivalently take the supremum over all* $0 \leq a < 1$*.*

Theorem 3.1. *An infinite sequence of real numbers* $(\alpha_j)_{j\in\mathbb{N}}$ *is uniformly distributed modulo 1 if and only if* $D_n^* \to 0$ *as* $n \to \infty$*, where* D_n^* *is the discrepancy modulo 1 of the first n terms of* $(\alpha_j)_{j\in\mathbb{N}}$*.*

Proof. If D is the discrepancy of the fractional parts of the numbers in S, we have that $D \leq D^*$ because of the property $\phi^*(a,b) = \phi(a,b)$ for $0 \leq a < b \leq 1$ and the definitions of D and D^*.
We also have $D^* \leq 2D$, since any interval $[a,b), 0 \leq a < 1$, $b - a \leq 1$ is the disjoint union of at most 2 intervals, each of which of the form $[a', b')$ for either $0 \leq a' < b' \leq 1$ or $1 \leq a' < b' \leq 2$. Thus:

$$\phi^*(a,b) = \sum \phi^*(a', b') \text{ , } b - a = \sum (b' - a'),$$

where the sum goes over at most two terms. Hence, we have the inequality

$$\left| \frac{\phi^*(a,b)}{Q} - (b-a) \right| \leq \sum \left| \frac{\phi^*(a',b')}{Q} - (b'-a') \right| \leq 2D,$$

and therefore $D^* \leq 2D$.
To sum up:

- we have shown that given a set of real numbers (α_j), $1 \leq j \leq Q$, the discrepancy D of the set of fractional parts $(\{\alpha_j\})_{1\leq j\leq Q}$ and the discrepancy modulo 1 D^* of S, are connected via: $D \leq D^* \leq 2D$;

- if $(\alpha_j)_{j\in\mathbb{N}}$ is a sequence of real numbers (not necessarily contained in $[0,1)$), D_n the discrepancy of the first n terms of $(\{\alpha_j\})_{j\in\mathbb{N}}$ and D_n^* the discrepancy modulo 1 of the first n terms, then the previous item implies that $D_n \to 0$ if and only if $D_n^* \to 0$ as $n \to \infty$.

In light of Theorem 2.1, this completes the proof. ☐

4 Weyl's theorems

Theorem 4.1. *Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of real numbers in $[0,1)$. Then $(\alpha_j)_{j\in\mathbb{N}}$ is uniformly distributed if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\alpha_k) = \int_0^1 f(x)\,dx \tag{6}$$

for every function f which is Riemann integrable in $[0,1]$.

Proof. We may assume f to be real-valued, otherwise we split f into real and imaginary parts and look at them separately.

($\Leftarrow$)

Let $[a,b)$ be an interval, where $0 \le a < b \le 1$, and define $f(x) = \begin{cases} 1 & x \in [a,b) \\ 0 & x \in [a,b)^c \end{cases}$ to be the characteristic function of the interval $[a,b)$. We then have

$$\frac{1}{n}\sum_{k=1}^{n} f(\alpha_k) = \frac{\phi_n(a,b)}{n} \text{ and } \int_0^1 f(x)\,dx = \int_a^b 1\,dx = b-a.$$

Therefore (6) implies that

$$\lim_{n\to\infty} \frac{\phi_n(a,b)}{n} = b-a,$$

which implies by Theorem 2.1, that the sequence $(\alpha_j)_{j\in\mathbb{N}}$ is uniformly distributed.

($\Rightarrow$)

Assume conversely that the sequence $(\alpha_j)_{j\in\mathbb{N}}$ is uniformly distributed. Then $\lim_{n\to\infty} \frac{\phi_n(a,b)}{n} = b-a$ holds, by Theorem 2.1, so that (6) holds for the characteristic function f of any interval $[a,b) \subset [0,1]$. Due to linearity (6) holds for any stepfunction in $[0,1]$. If f is Riemann integrable in $[0,1]$, then for every $\epsilon > 0$ one can find stepfunctions f_1, f_2, such that

$$f_1 \le f \le f_2 \text{ and } \int_0^1 (f_2(x) - f_1(x))\,dx < \epsilon.$$

Now (6) holds for f_1, thus

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f_1(\alpha_i) = \int_0^1 f_1(x)\,dx \geq \int_0^1 f(x)\,dx - \epsilon,$$

so for n large enough we have

$$\frac{1}{n}\sum_{i=1}^{n} f_1(\alpha_i) > \int_0^1 f(x)\,dx - 2\epsilon.$$

Similarly we get

$$\frac{1}{n}\sum_{i=1}^{n} f_1(\alpha_i) < \int_0^1 f(x)\,dx + 2\epsilon$$

for n large enough. Thus we obtain

$$\left|\frac{1}{n}\sum_{i=1}^{n} f(\alpha_i) - \int_0^1 f(x)\,dx\right| < 2\epsilon$$

for all n large enough, whence (6) is proved for all Riemann integrable functions in $[0,1]$. □

Theorem 4.2 (Weyl's criterion)**.** *Let $(\beta_j)_{j\in\mathbb{N}}$ be a sequence of real numbers. Then $(\beta_j)_{j\in\mathbb{N}}$ is uniformly distributed modulo 1 if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i m\beta_j} = 0$$

for all $m \in \mathbb{Z}\setminus\{0\}$.

Proof. (⇒)
Let $(\beta_j)_{j\in\mathbb{N}}$ be uniformly distributed modulo 1, and define $\alpha_j = \beta_j - [\beta_j] = \{\beta_j\}$ to be the fractional part of β_j. Then by definition $(\alpha_j)_{j\in\mathbb{N}}$ is uniformly distributed in $[0,1)$. Now taking $f(x) = e^{2\pi i m x}$, $m \in \mathbb{Z}\setminus\{0\}$ which is continous and hence Riemann integrable in Theorem 4.1, it follows that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i m\alpha_j} = \int_0^1 e^{2\pi i m x}\,dx = \frac{1}{2\pi i m}e^{2\pi i m} - \frac{1}{2\pi i m} = 0.$$

Now observe that $\beta_j - \alpha_j =: k_j \in \mathbb{Z}$. We therefore have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i m\beta_j} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i m\alpha_j}e^{2\pi i m k_j} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i m\alpha_j} = 0,$$

which completes the proof of ($\Rightarrow$).
($\Leftarrow$)
Suppose now that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i m \beta_j} = 0$$

for every $m \in \mathbb{Z} \setminus \{0\}$. This implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i m \alpha_j} = 0,$$

where $(\alpha_j)_{j\in\mathbb{N}}$ is defined as before.
Our goal is now to show that (6) holds for all Riemann integrable functions in $[0, 1]$. (6) obviously holds for $f(x) = 1$ and for $f(x) = e^{2\pi i m x}$, $m \in \mathbb{Z} \setminus \{0\}$ by our hypothesis. Hence it also holds by linearity for any trigonometric polynomial

$$a_0 + (a_1 \cos(2\pi x) + b_1 \sin(2\pi x)) + ... + (a_m \cos(2\pi x) + b_m \sin(2\pi x)), a_j, b_j \in \mathbb{C}\ \forall 1 \le j \le m.$$

Now we know by elementary Fourier analysis that any periodic function f of period 1 can be approximated uniformly by such a trigonometric polynomial to an arbitrary precision. Given $\epsilon > 0$, there exists a trigonometric polynomial f_ϵ such that:

$$\sup_{x\in[0,1]} |f(x) - f_\epsilon(x)| < \epsilon.$$

Now define $f_1 = f_\epsilon - \epsilon$ and $f_2 = f_\epsilon + \epsilon$, so that

$$f_1 \le f \le f_2 \text{ and } \int_0^1 (f_2(x) - f_1(x))\, dx = 2\epsilon.$$

As we have seen in the proof of Theorem 4.1, it follows that (6) holds for any continuous periodic function of period 1. If we now look at the unit interval $[0, 1]$, for any step function f in $[0, 1]$ we can find two periodic functions f_1 and f_2 such that

$$f_1 \le f \le f_2 \text{ and } \int_0^1 (f_2(x) - f_1(x))\, dx < \epsilon.$$

Hence (6) holds for any stepfunction in $[0, 1]$, which implies, as seen before, that it holds for any Riemann integrable function in $[0, 1]$. □

As an application of Theorem 4.2 we can state the following

Theorem 4.3. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$. Then the infinite sequence $(n\xi)$, $n = 1, 2, ...$ is uniformly distributed modulo 1.*

Proof. Let $m \in \mathbb{Z} \setminus \{0\}$, $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and define $\omega := m\xi$. We want to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i j \omega} = 0$$

Since $\xi \in \mathbb{R} \setminus \mathbb{Q}$, it follows that $\omega \notin \mathbb{Z}$ and therefore

$$\left| \sum_{j=1}^{n} e^{2\pi i j \omega} \right| = \left| \frac{1 - (e^{2\pi i \omega})^{n+1}}{1 - e^{2\pi i \omega}} - 1 \right| = \left| \frac{e^{2\pi i (n+1)\omega} - e^{2\pi i \omega}}{e^{2}\pi i \omega - 1} \right| \leq \frac{2}{|e^{2\pi i \omega} - 1|}.$$

Hence $\left| \sum_{j=1}^{m} e^{2\pi i j \omega} \right|$ is uniformly bounded in m, and therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i j \omega} = 0.$$

The desired result now follows by Theorem 4.2. □

5 Consequences and generalisations of Weyl's theorems

As we already mentioned, a uniformly distributed sequence in $[0, 1)$ is everywhere dense. From Theorem 4.3, we may thus deduce:

Corollary 5.1. *If ξ is an irrational number, then the sequence of fractional parts $(\{n\xi\})$, for $n = 1, 2, ...,$ is everywhere dense in the unit interval $[0, 1)$.*

The concept of uniform distribution can be easily generalized to spaces of dimension greater than one. Let $(P^{(j)})$ be an infinite sequence of points in a p-dimensional Euclidean space of the form $(x_{j1}, ..., x_{jp})$, where $p \geq 1$. Let α_{jr} denote the fractional part of x_{jr}, so that $0 \leq \alpha_{jr} < 1$ for $r = 1, ..., p$. If we denote by $\{P^{(j)}\}$ the vector of fractional parts $(\alpha_{j1}, ..., \alpha_{jp})$, then the point $\{P^{(j)}\}$ lies in the unit cube defined by the inequalities $0 \leq x_j < 1$ for $j = 1, ..., p$. Let V denote a rectangle (i.e. the cartesian product of p intervals of the form $[a, b)$) contained in the unit cube, and let $|V|$ denote its Lebesgue measure.

Definition. *We say that the infinite sequence $(P^{(j)})$ is uniformly distributed modulo 1 if and only if the corresponding sequence $(\{P^{(j)}\})$ is uniformly distributed in the unit cube, that is, if and only if*

$$\lim_{n\to\infty} \frac{\varphi_n(V)}{n} = |V|$$

for every rectangle V contained in the unit cube, where $\varphi_n(V)$ denotes the number of points among the first n terms of the sequence $(\{P^{(j)}\})$ which are contained in V.

Remark. *As in the one-dimensional case, this is equivalent to the statement*

$$\sup_V \left| \frac{\varphi_n(V)}{n} - |V| \right| \overset{n\to\infty}{\longrightarrow} 0$$

where the supremum is taken over all rectangles V as in the definition above.

Theorem 5.2. *The sequence $(\{P^{(j)}\})$ is uniformly distributed in the unit cube if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{h=1}^{n} e^{2\pi i[m_1\alpha_{h1}+\ldots+m_p\alpha_{hp}]} = 0$$

for every vector $(m_1, ..., m_p) \in \mathbb{Z}^p$, $(m_1, ..., m_p) \neq (0, ..., 0)$.

Proof. The proof here runs along the same lines as in the case of one variable (Theorem 4.2). We have only to observe that the integral of a step function can be approximated arbitrarily well from above and below by integrals of continuous function; in terms, continuous functions can be approximated uniformly by linear combinations of functions of the form $(n_1, ..., n_p) \mapsto e^{2\pi i(m_1 n_1, ..., m_p n_p)}$, $(m_1, ..., m_p) \in \mathbb{Z}^p$, by a classical result in Fourier analysis. □

Theorem 5.3. *If $\xi_1, ..., \xi_p$ are real numbers such that $\xi_1, ..., \xi_p, 1$ are linearly independent over the integers (i.e. there exists no linear relation of the form $\sum_{j=1}^{p} l_j \xi_j = l$, where l and l_j are integers and $(l_1, ..., l_p, l) \neq (0, ..., 0, 0)$), then the sequence $n\xi = (n\xi_1, ..., n\xi_p)$, where $n = 1, 2, ...,$ is uniformly distributed modulo 1.*

Proof. This generalisation of Theorem 4.3 is proven analogously via Theorem 5.2. □

6 Kronecker's theorem

Theorem 5.3 implies that the sequence $(\{n\xi\})$, where $\{n\xi\} = (\{n\xi_1\}, ..., \{n\xi_p\})$, is everywhere dense in the unit cube. This is known as Kronecker's theorem. We state it as:

Theorem 6.1 (Kronecker's theorem)**.** *If $\theta_1, ..., \theta_k, 1$ are real numbers linearly independent over the integers, $\alpha_1, ..., \alpha_k$ are arbitrary real numbers, and N and ε are positive real numbers, then there exist integers n and $p_1, ..., p_k$ such that*

$$n > N \quad \textit{and} \quad |n\theta_m - p_m - \alpha_m| < \varepsilon$$

for $m = 1, ..., k$.

We give another formulation of the previous theorem:

Theorem 6.2. *If $\theta_1, ..., \theta_k$ are real numbers linearly independent over the integers, $\alpha_1, ..., \alpha_k$ are arbitrary real numbers, and T and ε are positive real numbers, then there exist a real number t and integers $p_1, ..., p_k$ such that*

$$t > T \quad \text{and} \quad |t\theta_m - p_m - \alpha_m| < \varepsilon$$

for $m = 1, ..., k$.

Proposition. *Theorem 6.1 is equivalent Theorem 6.2.*

Proof. Assume Theorem 6.2 holds. It suffices to prove the implication with $0 < \theta_m \leq 1$ for $m = 1, ..., k$. That's because if $1, \theta_1, ..., \theta_k$ are linearly independent over the integers, so are $1, \theta'_1, ..., \theta'_k$, where $\theta'_j = \theta_j - q_j$, and q_j are suitable integers; the inequality $\left|n\theta'_m - p'_m - \alpha_m\right| < \varepsilon$, for an integer p'_m, implies that $|n\theta_j - p_m - \alpha_m| < \varepsilon$, where $p_m = p'_m + nq_m$. Let us then assume that $0 < \theta_m \leq 1$ for $m = 1, ..., k$, $0 < \varepsilon < 1$, and that $\theta_1, ..., \theta_k, 1$ are linearly independent over the integers. Then by Theorem 6.2, with $k+1$ instead of k, $N+1$ instead of T, and $\frac{1}{2}\varepsilon$ instead of ε, applied to the collections $\theta_1, ..., \theta_k, 1$ and $\alpha_1, ..., \alpha_k, 0$, there exist integers $p_1, ..., p_{k+1}$ and a real t with

$$t > N + 1 \quad , \quad |t\theta_m - p_m - \alpha_m| < \frac{1}{2}\varepsilon$$

for $m = 1, ..., k$ and

$$|t - p_{k+1}| < \frac{1}{2}\varepsilon \quad .$$

It follows that $p_{k+1} > t - \frac{1}{2}\varepsilon > N$, since $t > N+1$, and $\varepsilon < 1$. Since $0 < \theta_m \leq 1$, we have

$$|p_{k+1}\theta_m - p_m - \alpha_m| = |p_{k+1}\theta_m - t\theta_m + t\theta_m - p_m - \alpha_m| \leq$$

$$\leq |t\theta_m - p_m - \alpha_m| + |(p_{k+1} - t)\theta_m| \leq |t\theta_m - p_m - \alpha_m| + |p_{k+1} - t| < \varepsilon$$

for $m = 1, ..., k$. Thus Theorem 6.1 is proved with $n = p_{k+1}$.

Conversely, assume Theorem 6.1 is valid. If $k = 1$, Theorem 6.2 is trivial, so let us assume $k > 1$. Without loss of generality, we may assume $\theta_m > 0$. Let now $\theta_1, ..., \theta_k$ be linearly independent over the integers. Then the numbers $\frac{\theta_1}{\theta_k}, ..., \frac{\theta_{k-1}}{\theta_k}, 1$ are also linearly independent. If we apply Theorem 6.1, with $N = T\theta_k$, to the collections $\frac{\theta_1}{\theta_k}, ..., \frac{\theta_{k-1}}{\theta_k}$ and $\alpha_1, ..., \alpha_{k-1}$, it follows that there exist integers $p_1, ..., p_{k-1}$ and n with

$$n > N \quad , \quad \left|n\frac{\theta_m}{\theta_k} - p_m - \alpha_m\right| < \varepsilon$$

for $m = 1, ..., (k-1)$. If we now set $t = \frac{n}{\theta_k}$, then

$$t > T \quad , \quad |t\theta_m - p_m - \alpha_m| < \varepsilon$$

for $m = 1, ..., (k-1)$, while trivially

$$0 = |t\theta_k - n| < \varepsilon \quad ,$$

so that we have conclusion of Theorem 6.2 for the collections $\theta_1, ..., \theta_k$ and $\alpha_1, ..., \alpha_{k-1}, 0$. Similarly one can prove Theorem 6.2 for the sets $\theta_1, ..., \theta_k$ and $0, ..., 0, \alpha_k$. Together, these imply that Theorem 6.2 is valid for the collections of interest $\theta_1, ..., \theta_k$ and $\alpha_1, ..., \alpha_k$, for if the difference of $t\theta_m$ from α_m is nearly an integer, and the difference of $t'\theta_m$ from β_m is nearly an integer, then the difference of $(t+t')\theta_m$ from $\alpha_m + \beta_m$ is nearly an integer. Thus the equivalence of Theorem 6.1 and Theorem 6.2 is proved. □

We shall now give a proof of Theorem 6.2 due to Harald Bohr.

Proof of Theorem 6.2. Let

$$F(t) = 1 + \sum_{m=1}^{k} e^{2\pi i(t\theta_m - \alpha_m)} \tag{7}$$

where t is real. Then $0 \leq |F(t)| \leq k+1$.

If Theorem 6.2 is true, then for a sufficiently large t, every number $t\theta_m - \alpha_m$ is nearly an integer, and $|F(t)|$ is nearly $k+1$. Because if $x_m = t\theta_m - \alpha_m$, and $\varepsilon > 0$ is given, there exists a δ, such that if p_m is an integer and $|x_m - p_m| < \delta$, then $\left|e^{2\pi i x_m} - 1\right| < \varepsilon$.

Conversely, if $|F(t)|$ is nearly $k+1$ for some large t, then every term in the sum (7) must be nearly 1, since no term can exceed 1 in absolute value, and Theorem 6.2 must be true. This can be seen as follows. If there exists an $\eta \in (0,1)$, such that $|F(t)| \geq k+1-\eta$, and $z = e^{2\pi i x_m} = x + iy$, say, then it follows that $|y| \leq 2\sqrt{\eta}$. For

$$k + 1 - \eta \leq |F(t)| \leq (k-1) + \left|1 + e^{2\pi i x_m}\right|$$

or

$$2 \geq \left|1 + e^{2\pi i x_m}\right| \geq 2 - \eta \quad with \quad m = 1, ..., k$$

And

$$|1+z|^2 = (1+x)^2 + y^2 = (1+x)^2 + (1-x^2) = 2 + 2x \geq (2-\eta)^2 \geq 4 - 4\eta$$

so that $1 \geq x \geq 1 - 2\eta$. Now

$$y^2 = 1 - x^2 = (1-x)(1+x) \leq 2(1-x) \leq 4\eta$$

which implies that $|y| \leq 2\sqrt{\eta}$. Therefore $|z-1| < 4\sqrt{\eta}$

Claim: $\limsup_{t\to\infty} |F(t)| \geq k+1$

Proof. Since $|F(t)| \leq k+1$, to prove the claim it is sufficient to prove that

$$\limsup_{t\to\infty} |F(t)| < k+1 \tag{8}$$

which directly imply that there exist $\lambda \in \mathbb{R}$ such that $|F(t)| < \lambda < k+1$. Now let $\psi = \psi(x_1, ..., x_k) = 1 + x_1 + ... + x_k$ and p be a positive integer. Then

$$\psi^p = \sum_{\substack{n_1+...+n_k \leq p \\ n_j \geq 0, j=1,...k}} a_{n_1,...,n_k} x_1^{n_1} ... x_k^{n_k} \tag{9}$$

where the coefficients $a_{n_1,...,n_k}$ have the following properties:

- they are positive
- their sum $\sum a_{n_1,...,n_k} = \psi^p(1,1,...,1) = (k+1)^p$
- they are at most $(p+1)^k$ in number

We use this formalism to consider

$$F^p(t) = (1 + \sum_{m=1}^{k} e^{2\pi i (t\theta_m - \alpha_m)})^p$$

If we use (9) with $e^{2\pi i(t\theta_j - a_j)}$ in place of x_j, we see that $F^p(t)$ is a sum of the form

$$\sum b_\nu e^{c_\nu i t}$$

with $2\pi(n_1\theta_1 + ... + n_k\theta_k)$ taking the place of c_ν and $a_{n_1,...,n_k} e^{-2\pi i(n_1\alpha_1 + ... + n_k\alpha_k)}$ in the place of the b_ν. Since the θ's are linearly independent, the c_ν's are all different. Now, from the fact that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{cit}\, dt = \begin{cases} 0 & if \quad c \neq 0 \\ 1 & if \quad c = 0 \end{cases}$$

for some constant $c \in \mathbb{R}$, it consequently holds

$$b_\nu = \lim_{T\to\infty} \frac{1}{T} \int_0^T (F(t))^p e^{-c_\nu i t}\, dt$$

Now, since $|F(t)| < \lambda$, we have

$$a_{n_1,...,n_k} = |b_\nu| \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T |F(t)|^p\, dt \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \lambda^p\, dt = \lambda^p$$

This means that every coefficient in (9) satisfies the inequality $a_{n_1,\dots,n_k} \leq \lambda^p$. Since there are at most $(p+1)^k$ such coefficients, we have

$$(k+1)^p = \sum a_{n_1,\dots,n_k} \leq (p+1)^k \lambda^p$$

Since $\dfrac{\lambda}{k+1} < 1$ and $\displaystyle\lim_{p\to\infty} \left(\frac{\lambda}{k+1}\right)^p (p+1)^k = 0$, it follows that (8) is impossible, so that the claim is established. □

With this claim, the proof of the theorem is finalised. □