Student Seminar in Elementary Number Theory - Generating Functions of Arithmetical Functions

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1 Introduction

The goal of this presentation is to give an introduction to generating functions of arithmetical functions, which we have seen in the preceding weeks. Generating functions play an important role in the analytical interpretation of arithmetical functions. Analysing the generating function can sometimes give us more insight into number-theoretical concepts. A famous example is being able to show the existence of infinitely many primes using the zeta function, which we will see in more detail in the following presentation. We hope to give some motivation for the consideration of generating functions in number theory and provide insight into their importance.

2 Dirichlet series as generating functions

Given a sequence of numbers $\alpha_n \in \mathbb{C}$, any function of the form

$$F(s) = \sum_{n=1}^{\infty} \alpha_n u_n(s)$$

may be regarded as a generating function of α_n.

In general number theory can be divided into an 'additive' and a 'multiplicative' side, both of which we will touch on in this presentation. First we shall focus on the 'multiplicative' side and consider generating functions of the following form.

Definition 2.1. A Dirichlet series is any series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$$

where all $\alpha_n \in \mathbb{C}$.

Though the variable s may be real or complex, we shall only consider real values during this presentation.

Next we would like to present three theorems which are essential for our purpose. These are special cases of more general theorems, but still enough for our immediate purpose.

(1) If $\sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$ is absolutely convergent for a given $s \in \mathbb{R}$, then it is absolutely convergent for all greater s.

Proof. This follows directly from $|\alpha_n n^{-s_2}| \leq |\alpha_n n^{-s_1}|$ for $n \geq 1$ and $s_2 > s_1$. □

(2) If $\sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$ is absolutely convergent for $s > s_0$, the series may be differentiated term by term, so for $s > s_0$ it holds

$$F'(s) = -\sum_{n=1}^{\infty} \frac{\alpha_n \log n}{n^s}$$

Proof. Choose $\delta > 0, s_1, s_2 \in \mathbb{R}$ such that

$$s_0 < s_0 + \delta = s_1 \leq s \leq s_2$$

Then $\log n < K(\delta) n^{\frac{1}{2}\delta}$, where $K(\delta)$ depends only on δ and

$$\left| \frac{\alpha_n \log n}{n^s} \right| \leq K(\delta) \left| \frac{\alpha_n}{n^{s_0 + \frac{1}{2}\delta}} \right|$$

for all s between s_1 and s_2. The series

$$\sum_{n=1}^{\infty}\left|\frac{\alpha_n}{n^{s_0+\frac{1}{2}\delta}}\right|$$

is convergent, so it follows that the right side of the equality to be shown is uniformly convergent for all s between s_1 and s_2. This justifies the term by term differentiation. □

(3) If

$$F(s)=\sum_{n=1}^{\infty}\alpha_n n^{-s}=0$$

for all $s>s_0$, then $\alpha_n=0$ for all n.

Proof. By contradiction, we assume that there is a first non-zero coefficient α_m. Then it holds

$$0=F(s)=\sum_{n=m}^{\infty}\alpha_n n^{-s}=\alpha_m m^{-s}\Big(1+\frac{\alpha_{m+1}}{\alpha_m}\Big(\frac{m+1}{m}\Big)^{-s}+\frac{\alpha_{m+2}}{\alpha_m}\Big(\frac{m+2}{m}\Big)^{-s}+...\Big)$$

To lighten notation we define

$$G(s)=\sum_{n=m}^{\infty}\frac{\alpha_n}{\alpha_m}\Big(\frac{n}{m}\Big)^{-s}=\frac{\alpha_{m+1}}{\alpha_m}\Big(\frac{m+1}{m}\Big)^{-s}+\frac{\alpha_{m+2}}{\alpha_m}\Big(\frac{m+2}{m}\Big)^{-s}+...$$

If $s_0<s_1<s$, then

$$\Big(\frac{m+k}{m}\Big)^{-s}\leq\Big(\frac{m+1}{m}\Big)^{-(s-s_1)}\Big(\frac{m+k}{m}\Big)^{-s_1}$$

Using this, it follows

$$|G(s)|\leq\frac{1}{\alpha_n}\Big(\frac{m+1}{m}\Big)^{-(s-s_1)}m^{s_1}\sum_{n=m}^{\infty}\frac{|\alpha_{m+k}|}{(m+k)^{s_1}}$$

This converges to 0 when s goes to ∞, hence it has absolute value smaller than $\frac{1}{2}$ for sufficiently large s. Therefore we can use

$$|1+G(s)|\geq\frac{1}{2}$$

It follows that

$$0=|F(s)|=|\alpha_m m^{-s}(1+|G(s)|)|\geq|\alpha_m|m^{-s}\frac{1}{2}$$

for sufficiently large s and thus $\alpha_m=0$, giving the desired contradiction. □

A simple but important direct consequence of this is the following theorem, which we will need later on.

Theorem 2.2 (Uniqueness Theorem). *If*

$$\sum_{n=1}^{\infty}\alpha_n n^{-s}=\sum_{n=1}^{\infty}\beta_n n^{-s}$$

for all $s>s_1$, then $\alpha_n=\beta_n$ for all n.

3 The Zeta function

Definition 3.1. The simplest infinite Dirichlet series, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

is called the Riemann zeta function.

This series is convergent for $s > 1$.
Though it might look simple enough, there are many number-theoretical results that can be proven using the zeta function, and regarding this function is the most famous unsolved problem in mathematics, the Riemann Hypothesis, which we will see later on in the presentation.

Since the series is absolutely convergent for $s > 1$, we may differentiate term by term to obtain

Theorem 3.2.

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

The aforementioned importance of the zeta function in the theory of prime numbers relies on the following remarkable identity discovered by Euler.

Theorem 3.3 (Euler's Identity)**.** *If $s > 1$, then*

$$\zeta(s) = \prod_{p\,prime} \frac{1}{1-p^{-s}}$$

Proof. Since $p \geq 2$, the factor $\frac{1}{1-p^{-s}}$ can be rewritten as the geometric series

$$\frac{1}{1-p^{-s}} = \sum_{k=0}^{\infty} (p^{-s})^k = 1 + p^{-s} + p^{-2s} + ...$$

for $s > 1$. If we take the first prime numbers $p = 2, 3, 5, ..., P$ and multiply together the series given above, we get the general term

$$2^{-a_2 s} 3^{-a_3 s} ... P^{-a_P s} = n^{-s}$$

where n defined as $n = 2^{a_2} 3^{a_3} ... P^{a_P}$ only occurs if it has no prime factors greater than P, and then once only by the Fundamental Theorem of Arithmetic. So it follows that

$$\prod_{p \leq P} \frac{1}{1-p^{-s}} = \sum_{(P)} n^{-s},$$

the term on the right hand denoting the sum over all numbers with all prime factors smaller or equal P. It is obvious that all numbers up to P are included in this sum, so the following inequalities hold:

$$0 < \sum_{n=1}^{\infty} n^{-s} - \sum_{(P)} n^{-s} < \sum_{n=P+1}^{\infty} n^{-s}$$

The last sum tends to zero as P goes to infinity, so we can conclude the proof with

$$\sum_{n=1}^{\infty} n^{-s} = \lim_{P\to\infty} \sum_{(P)} n^{-s} = \lim_{P\to\infty} \prod_{p\le P} \frac{1}{1-p^{-s}}$$

□

4 The behaviour of $\zeta(s)$ as $s \to 1$

Below we state a few theorems that will become essential later in the presentation, since we will require to know how $\zeta(s)$ and $\zeta'(s)$ behave as s tends to 1 through values greater than 1. For this purpose we rewrite $\zeta(s)$ in the form

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_1^{\infty} x^{-s}\,dt + \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s})\,dx$$

Since $s > 1$, we have

$$\int_1^{\infty} x^{-s}\,dt = \frac{1}{s-1}$$

and

$$0 < n^{-s} - x^{-s} < \frac{s}{n^2}$$

if $n < x < n+1$. The last inequality follows from the Fundamental Theorem of calculus, since

$$n^{-s} - x^{-s} = \int_n^x st^{-s-1}\,dt$$

for $n < x$ and

$$\int_n^x t^{-s-1}\,dt < \int_n^x t^{-2}\,dt < \int_n^x n^{-2}\,dt < \frac{1}{n^2},$$

using $s > 1$ for the first, $n < t$ for the second and $n < x < n+1$ for the third inequality.

Hence

$$0 < \int_n^{n+1} (n^{-s} - x^{-s})\,dx < \frac{s}{n^2}$$

and

$$0 < \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s})\,dx < s \sum_{n=1}^{\infty} \frac{1}{n^2}$$

From the inequalities above we can derive the following theorems:

Theorem 4.1.

$$\zeta(s) = \frac{1}{s-1} + O(1)$$

Theorem 4.2.

$$\log \zeta(s) = \log \frac{1}{s-1} + O(s-1)$$

Theorem 4.2 follows from Theorem 4.1 using

$$\log \zeta(s) = \log \frac{1}{s-1} + \log(1 + O(s-1))$$

As to the function's derivative, we may proceed analogously writing

$$\begin{aligned} -\zeta'(s) &= \sum_{n=1}^{\infty} n^{-s} \log n \\ &= \int_1^{\infty} x^{-s} \log x \, dx + \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} \log n - x^{-s} \log x) \, dx \end{aligned}$$

to deduce

Theorem 4.3.

$$\zeta'(s) = -\frac{1}{(s-1)^2} + O(1)$$

In particular we obtain

$$\zeta(s) \sim \frac{1}{s-1}.$$

5 Euler's proof of the infinitude of primes

As promised, we can now show that there are infinitely many primes using the zeta function. We consider $s = 1$, in which case Euler's Identity does not hold since $\mathrm{Re}(s) > 1$ is not fulfilled, so we must consider the limit as $s \to 1$.

For $s \to 1$ we now know that

$$\zeta(s) \sim \frac{1}{s-1}$$

and

$$\log \zeta(s) \sim \log \frac{1}{s-1}.$$

Using Euler's Identity for $\mathrm{Re}(s) > 1$ we get

$$\begin{aligned} \log(\zeta(s)) &= \log \left(\prod_{p\, prime} \frac{1}{1-p^{-s}} \right) = \sum_{p\, prime} \log \left(\frac{1}{1-p^{-s}} \right) \\ &= \sum_{p\, prime} \sum_{n=1}^{\infty} \frac{p^{-sn}}{n} = \sum_{p\, prime} p^{-s} + \sum_{p\, prime} \sum_{n=2}^{\infty} \frac{p^{-sn}}{n}. \end{aligned}$$

Here we use the expansion $\log(1-x)^{-1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x| < 1$ for the third equality and in the last equality we part the sum by considering $n = 1$ separately.

We bound the difference of $\log(\zeta(s))$ and $\sum_{p\, prime} p^{-s}$ from above to analyze the behaviour of the sum for $s \to 1$.

$$\begin{aligned}\sum_{p\,prime}\sum_{n=2}^{\infty}\frac{p^{-sn}}{n} &< \sum_{p\,prime}\sum_{n=2}^{\infty}p^{-sn}\\ &= \sum_{p\,prime} p^{-2s}\sum_{n=0}^{\infty}p^{-sn}\\ &= \sum_{p\,prime} p^{-2s}\frac{1}{1-p^{-s}}\\ &= \sum_{p\,prime}\frac{1}{p^s(p^s-1)}\\ &< \sum_{n=2}^{\infty}\frac{1}{n^s(n^s-1)}\\ &< \sum_{n=2}^{\infty}\frac{1}{n(n-1)} = 1\end{aligned}$$

In the third step we used the fact that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

Above we have shown $\left|\log(\zeta(s)) - \sum_{p\,prime} p^{-s}\right| < 1$ so it follows as $s \to 1$ from above

$$\sum_{p\,prime} p^{-s} \sim \log(\zeta(s)) \sim \log\left(\frac{1}{1-s}\right)$$

meaning that $\sum_{p\,prime} p^{-s}$ diverges as $s \to 1$, so the sum must be an infinite sum, proving that there is an infinite number of primes.

This is an interesting example showing us that generating functions can let us derive important number-theoretical results from a seemingly simple function. The zeta function does not seem to have anything to do with primes at first glance, but can be used to show one of their most important properties. Though this function seems harmless at first, it poses a problem considered by many to be the most important unsolved problem in pure mathematics. The Riemann Hypothesis is one of the Millennium Problems and we would like to give an overview of the problem and motivate its importance in the following part of the presentation.

6 The Riemann Hypothesis

The zeta function, though originally defined on $\text{Re}(s) > 1$, admits a meromorphic extension to $\mathbb{C}$ with a single pole at $s = 1$. In this part of the presentation we will allow s to take complex values. The Riemann Hypothesis is the conjecture that all non-trivial zeros of the zeta function are on the line

$$\left\{s \in \mathbb{C} | \text{Re}(s) = \tfrac{1}{2}\right\}. \qquad \textit{section}$$

Euler's identity, which we saw before, is true for all s with real part larger than one, so we can deduce that the zeta function is non vanishing for $\text{Re}(s) > 1$. We can show that an infinite product $\prod_{n=1}^{\infty} a_n$ converges to a nonzero real number if and only if the series $\sum n = 1^{\infty} \log a_n$

converges. Thus the infinite product $\prod_{n=1}^{\infty} \frac{1}{1-p^{-s}}$ does not converge, since

$$\frac{1}{1-p^{-s}} = \frac{p^s}{p^s-1} > 1$$

for all p prime and $s > 1$ and so $\sum n = 1^{\infty} \log \frac{1}{1-p^{-s}}$ diverges. Then it follows from Euler's Identity that the zeta function is non vanishing for $\mathrm{Re}(s) > 1$. The term non-trivial zeros used in the hypothesis easily explained by looking at the trivial zeros in comparison. The zeta function has trivial zeros at all even negative integers. This can be seen in a functional equation relating $\zeta(s)$ to $\zeta(1-s)$. An in depth understanding of the relation is not very essential for our purpose, we only look at it here to verify these trivial zeros. It uses the gamma function we have already seen in the Analysis I course, the definition given below.

Definition 6.1. The gamma function is given by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt.$$

One characteristic of this function is that

$$\Gamma(n) = (n-1)^{-1}$$

for any natural number n. Using this function, the relation between $\zeta(s)$ and $\zeta(1-s)$ is given by

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \quad \forall s \in \mathbb{C}$$

So if we choose $s = -2k$ for k a strictly positive natural number, we observe that

$$\begin{aligned}\zeta(-2k) &= 2^{-2k}\pi^{-2k-1} \sin\left(\frac{\pi(-2k)}{2}\right)\Gamma(1+2k)\zeta(1+2k) \\ &= 2^{-2k}\pi^{-2k-1} \sin(-\pi k) 2k! \zeta(1+2k) \\ &= 0.\end{aligned}$$

Thus we observe that the zeta function has zeros at the negative even integers and from the functional equation from above it also follows that there are no other zeros in the area $\mathrm{Re}(s) < 0$, since the terms are non-vanishing. Hence we can deduce that all non-trivial zeros of the function must lie in the area $\mathrm{Re}(s) \in [0, 1]$, which is called the critical strip. One can also show that the zeta function is non-vanishing on $\mathrm{Re}(s) = 1$, though the proof will not be covered in this presentation. This fact already has an immensely important implication, the Prime Number Theorem stated here.

Theorem 6.2.

$$\frac{\pi(x)}{\frac{x}{\log x}} \xrightarrow[n\to\infty]{} 1$$

where $\pi(x)$ counts the number of primes less than or equal to x.

Though we will not prove the Prime Number Theorem in this presentation, it is another example of how important the zeta function is for number theory.

After all of these observations, we have reached the conclusion that all non-trivial zeros of the zeta function lie in the area $\mathrm{Re}(s) \in [0, 1)$. This is where the Riemann Hypothesis goes even further, claiming that these zeros are only on $\mathrm{Re}(s) = \frac{1}{2}$, which we call the critical line. There is an infinite number of roots on the critical line and proving that they are the zeta function's only non-trivial zeros would have far reaching consequences in number theory.

7 Multiplication of Dirichlet series

To motivate the multiplication of Dirichlet series, we recall the Dirichlet product introduced in a previous presentation.

Definition 7.1. The Dirichlet product of two arithmetical functions f and g is defined as

$$(f * g)(n) = \sum_{d|n} f(d) g\Big(\frac{n}{d}\Big).$$

Now suppose we are given a finite set of Dirichlet series

$$\sum \alpha_n n^{-s}, \sum \beta_n n^{-s}, \sum \gamma_n n^{-s}, ...$$

we shall multiply them together in the sense of forming all possible products with one factor selected from each series. We do this by defining the formal product as below.

Definition 7.2. The formal product of the finite set of Dirichlet series as above is defined as $\sum \chi_n n^{-s}$ where

$$\chi_n = \sum_{uvw...=n} \alpha_u \beta_v \gamma_w \cdots$$

In the case where we consider the formal product of only two Dirichlet series $\sum \alpha_n n^{-s}$ and $\sum \beta_n n^{-s}$, we denote their formal product by $\sum \gamma_n n^{-s}$ with

$$\gamma_n = \sum_{uv=n} \alpha_u \beta_v = \sum_{d|n} \alpha_d \beta_{\frac{n}{d}} = \sum_{d|n} \alpha_{\frac{n}{d}} \beta_d \tag{1}$$

wherein the similarity to the Dirichlet product of arithmetical functions can be seen. The formal product defined here becomes concrete when both series are absolutely convergent, as stated in the following theorem.

Theorem 7.3. *If the series $F(s) = \sum \alpha_u u^{-s}$ and $G(s) = \sum \alpha_v v^{-s}$ are absolutely convergent, then*

$$F(s)G(s) = \sum \gamma_n n^{-s}$$

where γ_n is defined as above.

This theorem follows from the fact that the absolute convergence of the series allows us to do arithmetical operations and rearrange the product into this form.

Using the Uniqueness Theorem we considered at the beginning of this presentation, we can also deduce that if

$$H(s) = \sum \delta_n n^{-s} = F(s)G(s)$$

then $\delta_n = \gamma_n$ for all n.

The definition of the formal product may be extended with some care to an infinite set of Dirichlet series. It is convenient to suppose

$$\alpha_1 = \beta_1 = \gamma_1 = ... = 1$$

since then the term

$$\alpha_u \beta_v \gamma_w ...$$

in our definition of χ_n will only include a finite number of factors which are not 1. This follows from the fact that we have an infinite number of factors $uvw...$ whose product is n, meaning that only a finite number of the factors may be different from 1.

We consider the most important case in the following theorem.

Theorem 7.4. *If $f(1) = 1$, $f(n)$ is multiplicative and*

$$\sum_{n=1}^{\infty} |f(n)| n^{-s}$$

it convergent, then

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s} = \prod_{p\, prime} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + ...).$$

The absolute convergence of

$$F_p(s) = 1 + f(p)p^{-s} + f(p^2)p^{-2s} + ...$$

is a corollary of the convergence of $\sum_{n=1}^{\infty} |f(n)| n^{-s}$. Thus we obtain, similarly to the proof of the Euler's Identity for the zeta function we have seen before,

$$\prod_{p \le P} F_p(s) = \sum_{(P)} |f(n)| n^{-s} = \sum_{n=1}^{\infty} |f(n)| n^{-s} - \sum_{(P)} |f(n)| n^{-s} \le \sum_{n=P+1}^{\infty} |f(n)| n^{-s} \to 0.$$

8 The generating functions of some special arithmetical functions

Most of the arithmetical functions we have considered in the previous weeks have generating functions which are simple combinations of the zeta function. We would like to remind you of some of these arithmetical functions and show some of the most important examples.

Recall the Möbius function, which was introduced in a previous presentation and will be analysed in even more detail further on.

Definition 8.1. The Möbius function $\mu(n)$ is defined as follows:

(1) $\mu(1) = 1$
(2) $\mu(n) = 0$ if n has a square factor
(3) $\mu(p_1 p_2 ... p_k) = (-1)^k$ if all the primes $p_1, p_2, ..., p_k$ are different

Theorem 8.2.

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $s > 1$.

Proof. This theorem is a consequence of Theorem 3.3 and Theorem 7.4, since

$$\begin{aligned}\frac{1}{\zeta(s)} &= \prod_{p\,prime} (1-p^{-s}) \\ &= \prod_{prime} (1+\mu(p)p^{-s}+\mu(p^2)p^{-2s}+...) \\ &= \sum_{n=1}^{\infty} \mu(n)n^{-s}.\end{aligned}$$

□

For the next theorem, we recall Euler's totient function $\phi(n)$ defined as the number of positive integers less than n and prime to it. It was proven in a previous presentation to be equal to

$$\phi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right). \tag{2}$$

Theorem 8.3.

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}$$

for $s > 2$.

Proof. This follows from Theorem 8.2, Theorem 7.3 and

$$\begin{aligned}\frac{\zeta(s-1)}{\zeta(s)} &= \sum_{n=1}^{\infty}\frac{n}{n^s}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \\ &= \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{d|n} d\mu\left(\frac{n}{d}\right) \\ &= \sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}\end{aligned}$$

The last equality follows from the the formula $\phi(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right)$, proven in a previous presentation. □

The next theorems deal with the functions $d(n)$ and $\sigma_k(n)$. Recall that $d(n)$ is defined as the number of divisors of n including 1 and n, while $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n. In particular $d(n)$ is equal to $\sigma_0(n)$, obviously.

Theorem 8.4.

$$\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^s}$$

for $s > k+1$.

Proof.

$$\begin{aligned}
\zeta(s)\zeta(s-k) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n=1}^{\infty} \frac{n^k}{n^s} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} d^k \\
&= \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}
\end{aligned}$$

Here we use Theorem 7.3 for the second equality. □

This theorem implies the following two special cases for $k = 0$ and $k = 1$.

Theorem 8.5.

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

for $s > 1$.

Theorem 8.6.

$$\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n^s}$$

for $s > 2$.

9 Deriving the Möbius inversion Formula

In this second part of the presentation we will focus on using some of the theorems we have talked about before. Specifically we will look at certain arithmetical functions which were discussed in past presentations of this course.

Recall the following theorem from a previous presentation:

Theorem 9.1. *For two arithmetical functions $f(n)$ and $g(n)$ which fulfill the condition*

$$g(n) = \sum_{d|n} f(d)$$

the Möbius inversion formula holds:

$$f(n) = \sum_{d|n} \mu(\frac{n}{d})g(d) = \sum_{d|n} \mu(d)g(\frac{n}{d}),$$

where $\mu(n)$ is the Möbius Function defined above (Definition 8.1).

We can prove a special case of Theorem 9.1 using generating functions. For this, assume that

$$g(n) = \sum_{d|n} f(d)$$

We denote $F(s)$ and $G(s)$ to be the generating functions of $f(n)$ and $g(n)$, using the Dirichlet series. Then, if these series are absolutely convergent, we have

$$\begin{aligned} F(s) \cdot \zeta(s) &= \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} f(d) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \\ &= G(s), \end{aligned}$$

where first we insert the definitions of $F(s)$ and $\zeta(s)$ and then use the rule for the formal product of two Dirichlet series, as introduced before in Theorem 7.3. Lastly we use the assumption relating $f(n)$ and $g(n)$.

Theorem 8.2 gives us an identity of the zeta Function $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$. Hence we get

$$F(s) = \frac{G(s)}{\zeta(s)} = \left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s},$$

where, again using the product of two Dirichlet series, (see Theorem 7.3)

$$h(n) = \sum_{d|n} g(d)\mu(\frac{n}{d}).$$

Using the Uniqueness Theorem (Theorem 2.2), we have

$$f(n) = h(n) = \sum_{d|n} g(d)\mu(\frac{n}{d}),$$

which now gives us exactly the Möbius inversion formula. But note that we cannot take this chain of arguments as a full proof, because in the beginning we assume that $F(s)$ and $G(s)$

converge absolutely. But according to the proof in one of the past presentations, this is not a necessary condition for Theorem 9.1 to hold.

10 The generating function of $\Lambda(n)$

The arithmetical function $\Lambda(n)$ is especially important in the analytical theory of primes, and is defined to be the logarithm $\log(p)$ when n is some power of a prime number p and 0 otherwise. More precisely:

(i) $\Lambda(n) = \log(p)$ if $n = p^m$ for some prime p and $m \in \mathbb{N}$,
(ii) $\Lambda(n) = 0$ otherwise.

This means that e.g. $\Lambda(8) = \Lambda(2^3)$ =log(2) or also that $\Lambda(9) = \Lambda(3^2)$ =log(3) and similarly $\Lambda(12) = \Lambda(2^2 \cdot 3) = 0$.
We can check whether $\Lambda(n)$ is multiplicative. For this, recall that an arithmetical function $f(n)$ is multiplicative if $f(m \cdot n) = f(m) \cdot f(n)$ for $n, m \in \mathbb{N}$ coprime. But notice that $\Lambda(2 \cdot 3) = 0$ and $\Lambda(2) \cdot \Lambda(3) = \log(2) \cdot \log(3) \neq 0$. Thus $\Lambda(n)$ is not multiplicative.

Theorem 10.1. *For all $s \in \mathbb{R}$ with $s > 1$, the generating function of $\Lambda(n)$ is given by*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Proof. We know from Theorem 3.3 that

$$\zeta(s) = \prod_{p\, prime} \frac{1}{1 - p^{-s}}$$

Taking the logarithm on both sides gives us

$$\log(\zeta(s)) = \sum_{p\, prime} \log\left(\frac{1}{1 - p^{-s}}\right)$$

Now we can differentiate on both sides with respect to s and then multiply by -1. We can do this, because the series above is uniformly convergent for $s \geq 1 + \delta > 1$. This gives us for the left-hand side

$$-\frac{d}{ds}\log(\zeta(s)) = -\frac{\zeta'(s)}{\zeta(s)}$$

and for each summand on the right hand side

$$\begin{aligned} -\frac{d}{ds}\log\left(\frac{1}{1 - p^{-s}}\right) &= -\frac{d}{ds}\log\left((1 - p^{-s})^{-1}\right) = \frac{d}{ds}\log\left(1 - p^{-s}\right) \\ &= \frac{1}{1 - p^{-s}} \cdot (\log(p) \cdot p^{-s}) = \frac{\log(p)}{p^s - 1} \end{aligned}$$

So we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p\, prime} \frac{\log(p)}{p^s - 1} \tag{10.1.1}$$

Now we want to write $\frac{1}{p^s - 1}$ differently by using the fact that $s > 1$ and $p \geq 2$, which implies that $0 < p^{-s} < 1$. We get the following

$$\begin{aligned}
\frac{1}{p^s - 1} &= p^{-s}\frac{1}{1-p^{-s}} \\
&= p^{-s} \cdot \left(1 + (p^{-s})^1 + (p^{-s})^2 + (p^{-s})^3 + ...\right) \\
&= p^{-s} + p^{-2s} + p^{-3s} + p^{-4s} + ... \\
&= \sum_{m=1}^{\infty} p^{-ms}
\end{aligned}$$

Inserting this into equation (10.1.1), gives

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p\, prime} \log(p) \sum_{m=1}^{\infty} p^{-ms} = \sum_{\substack{p\, prime \\ m\in\mathbb{N}}} \log(p)p^{-ms} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s},$$

where we can bring $\log(p)$ inside the inner sum because $\sum_{\substack{p\, prime \\ m\in\mathbb{N}}} \log(p)p^{-ms}$ is absolutely convergent when $s > 1$. In the last step we use the definition the the function $\Lambda(n)$.

□

In the sequel we will prove two theorems which relate $\Lambda(n)$ and $\log(n)$.

Theorem 10.2. *We have that*

$$\Lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log(d).$$

Proof. Thanks to Theorem 3.2 we know that $\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log(n)}{n^s}$ for $s > 1$. Let us now apply this together with Theorem 10.1:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} &= \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{\log(n)}{n^s} \\
&= \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{\log(n)}{n^s}\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{d|n} \mu\left(\frac{n}{d}\right) \log(d)\right)
\end{aligned}$$

Like in previous proofs we used the formula for the product of two Dirichlet series. Now by the Uniqueness Theorem (Theorem 2.2) we obtain the desired equality

$$\Lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log(d).$$

□

Theorem 10.3. *The following holds true:*

$$\log(n) = \sum_{d|n} \Lambda(d)$$

Proof. We use the same identities for the zeta function $\zeta(s)$ as in the previous proof and apply them in conjunction with Theorem 10.1. Doing this gives, for $s > 1$,

$$-\zeta'(s) = \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\log(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{d|n} 1 \cdot \Lambda(d)\right)$$

Then again by the Uniqueness Theorem (Theorem 2.2), we deduce

$$\log(n) = \sum_{d|n} \Lambda(d)$$

□

Alternative Proof of Theorem 10.3. We can also prove this theorem directly. For this we assume $n = \prod_{p\,prime} p^{a_p}$ where $a_p \in \mathbb{Z}_{\geq 0}$. Then by definition of $\Lambda(n)$ and properties of the logarithm we have that

$$\sum_{d|n} \Lambda(d) = \sum_{\substack{p^\alpha | n \\ p\,prime \\ \alpha \in \mathbb{N}}} \log(p) = \sum_{p\,prime} a_p \log(p) = \log\left(\prod_{p\,prime} p^{a_p}\right) = \log(n),$$

which gives us the statement of the theorem. □

Theorem 10.4. *For the Möbius function $\mu(n)$ it holds that*

$$-\mu(n)\log(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \Lambda(d)$$

Proof. Regard the following expression:

$$-\frac{d}{ds}\left[\frac{1}{\zeta(s)}\right] = \frac{\zeta'(s)}{\zeta^2(s)} = -\frac{1}{\zeta(s)} \cdot \left(-\frac{\zeta'(s)}{\zeta(s)}\right) = -\frac{1}{\zeta(s)} \cdot \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right)$$

$$= -\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right) = -\sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{d|n} \mu\left(\frac{n}{d}\right) \Lambda(d)\right).$$

Also, differentiating term by term,

$$-\frac{d}{ds}\left[\frac{1}{\zeta(s)}\right] = -\frac{d}{ds}\left[\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right] = \sum_{n=1}^{\infty} \frac{\mu(n) \cdot \log(n)}{n^s}$$

Equality of the last two expressions gives us

$$-\sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{d|n} \mu\left(\frac{n}{d}\right) \Lambda(d)\right) = \sum_{n=1}^{\infty} \frac{\mu(n) \cdot \log(n)}{n^s}$$

So by the Uniqueness Theorem (Theorem 2.2) we deduce

$$-\mu(n)\log(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \Lambda(d).$$

□

Theorem 10.5. *It holds*

$$\Lambda(n) = -\sum_{d|n} \mu(d)\log(d)$$

Proof. From previous presentations, recall the following implication: $f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) \Rightarrow g(n) = \sum_{d|n} f(d)$. We can apply this together with Theorem 10.4. This means applying the implication to the term

$$-\mu(n)\log(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\Lambda(d),$$

which gives us exactly

$$\Lambda(n) = -\sum_{d|n} \mu(d)\log(d)$$

Note: This theorem can also be proven by handling the term $-\frac{\zeta'(s)}{\zeta(s)} = \zeta(s) \cdot \frac{d}{ds}\left[\frac{1}{\zeta(s)}\right]$ like in the proof of Theorem 10.4. ☐

11 Some more examples of generating functions

In this Section we will give further examples of generating functions of certain arithmetical functions, some of which we have seen before.

Theorem 11.1. *Let $\lambda(n) = (-1)^\rho$ where ρ is the total number of prime factors of n (counted with their multiplicity). Its generating function for $s > 1$ is given by*

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$$

Proof. The theorem follows from the identity $\zeta(s) = \prod_{p\,prime} \frac{1}{1-p^{-s}}$:

$$\begin{aligned}
\frac{\zeta(2s)}{\zeta(s)} &= \prod_{p\,prime} \frac{1-p^{-s}}{1-p^{-2s}} = \prod_{p\,prime} \frac{1-p^{-s}}{(1-p^{-s})\cdot(1+p^{-s})} \\
&= \prod_{p\,prime} \frac{1}{1-(-p^{-s})} = \prod_{p\,prime} \left(1 + (-p^{-s})^1 + (-p^{-s})^2 + (-p^{-s})^3 + ...\right) \\
&= \prod_{p\,prime} \left(1 - \frac{1}{p^s} + \frac{1}{p^{2s}} - \frac{1}{p^{3s}} + ...\right) \\
&= \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.
\end{aligned}$$

The last step follows from taking $f(n) = \lambda(n)$ for $f(n)$ in Theorem 7.4. ☐

Theorem 11.2. *Let $q(n) = 1$ if n has no square factor and $q(n) = 0$ otherwise, so that $q(n) = |\mu(n)|$ where $\mu(n)$ is the Möbius function. The generating function of $q(n)$ for $s > 1$ is given by*

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{q(n)}{n^s} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}$$

Proof. The theorem follows using again the identity $\zeta(s) = \prod_{p\,prime} \frac{1}{1-p^{-s}}$:

$$\begin{aligned}
\frac{\zeta(s)}{\zeta(2s)} &= \prod_{p\,prime} \frac{1-p^{-2s}}{1-p^{-s}} = \prod_{p\,prime} \frac{(1-p^{-s})\cdot(1+p^{-s})}{1-p^{-s}} \\
&= \prod_{p\,prime} (1+p^{-s}) \\
&= \sum_{n=1}^{\infty} \frac{q(n)}{n^s} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}.
\end{aligned}$$

Like in the proof of Theorem 11.1, the second-to-last step follows from Theorem 7.4. □

Theorem 11.3. *Consider the arithmetical function $d(n)$ which gives the number of divisors of n, including 1 and n. The generating function of $(d(n))^2$ is given by*

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{(d(n))^2}{n^s} \quad (s > 1).$$

Proof. Like in previous proofs, we use the identity $\zeta(s) = \prod_{p\,prime} \frac{1}{1-p^{-s}}$. This gives us

$$\begin{aligned}
\frac{\zeta^4(s)}{\zeta(2s)} &= \prod_{p\,prime} \frac{1-p^{-2s}}{(1-p^{-s})^4} = \prod_{p\,prime} \frac{(1-p^{-s})\cdot(1+p^{-s})}{(1-p^{-s})^4} \\
&= \prod_{p\,prime} \frac{(1+p^{-s})}{(1-p^{-s})^3}
\end{aligned}$$

Notice that we can write

$$\begin{aligned}
\frac{1+x}{(1-x)^3} &= (1+x)\left(1+x+x^2+x^3+...\right)^3 \\
&= (1+x)\sum_{N=1}^{\infty} x^N \left(\sum_{\substack{n+m+k=N \\ n,m,k\in\mathbb{Z}_{\geq 0}}} 1 \right) \\
&= (1+x)\left(1+3x+6x^2+...\right) = 1+4x+9x^2+... \\
&= \sum_{l=0}^{\infty} (l+1)^2 x^l
\end{aligned}$$

Thus we get

$$\frac{\zeta^4(s)}{\zeta(2s)} = \prod_{p\,prime} \left(\sum_{l=0}^{\infty} (l+1)^2 p^{-ls} \right).$$

Note that when $n = p_1^{l_1} p_2^{l_2} p_3^{l_3} ... p_k^{l_k}$, we have by the definition of $d(n)$ that

$$(d(n))^2 = (l_1+1)^2(l_2+1)^2(l_3+1)^2...(l_k+1)^2,$$

so again by Theorem 7.4 we obtain

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{(d(n))^2}{n^s} \quad (s > 1).$$

□

12 The generating function of $r(n)$

We define $r(n)$ as the number of ways to write n as a sum of two squares $n = n_1^2 + n_2^2$ where $n_1, n_2 \in \mathbb{Z}$.
For example $r(1) = 4$, because $1 = (\pm 1)^2 + 0^2 = 0^2 + (\pm 1)^2$.
We have that

$$r(n) = 4 \sum_{d|n} \chi(d)$$

Where $\chi(n) = 0$ when n is even and $\chi(n) = (-1)^{\frac{1}{2}(n-1)}$ when n is odd.

Theorem 12.1. *The generating function of $r(n)$ is given by*

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = 4\zeta(s)L(s) \quad (s > 1),$$

where $L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = 1^{-s} - 3^{-s} + 5^{-s} - ...$

Proof. This follows directly from the definition of multiplication of two Dirichlet series:

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(4 \sum_{d|n} 1 \cdot \chi(d) \right) = 4 \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right) \cdot \left(\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \right) = 4\zeta(s)L(s).$$

□

Remark 12.2. Note that the function $\eta(s) = 1^{-s} - 2^{-s} + 3^{-s} - ...$ can be expressed in terms of $\zeta(s)$ by using the formula

$$\eta(s) = (1 - 2^{1-s})\zeta(s).$$

Further note that by Theorem 7.4, $L(s)$ can be written as

$$\begin{aligned} L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} &= \prod_{p\,prime} \left(1 + \chi(p)p^{-s} + \chi(p^2)p^{-2s} + \chi(p^3)p^{-3s} + ...\right) \\ &= \prod_{p\,prime} \left(1 + \chi(p)p^{-s} + (\chi(p))^2 p^{-2s} + (\chi(p))^3 p^{-3s} + ...\right) \\ &= \prod_{p\,prime} \left(1 + (\chi(p)p^{-s})^1 + (\chi(p)p^{-s})^2 + (\chi(p)p^{-s})^3 + ...\right) \\ &= \prod_{p\,prime} \left(\frac{1}{1 - \chi(p)p^{-s}}\right) \end{aligned}$$

Note that we used the following property of $\chi(n)$: If $n, m \in \mathbb{N}$ are both odd, then $\chi(n \cdot m) = \chi(n) \cdot \chi(m)$.
This is the basis of the analytical theory of the distribution of primes of the form $4m+1$ and $4m+3$.

13 Different types of generating functions

As you might remember from the beginning of our presentation, the Dirichlet series of the form $\sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$ are not the only kind of generating functions. Any function of the form

$$F(s) = \sum_{n=1}^{\infty} \alpha_n u_n(s)$$

can be called a generating function of α_n. Today we have focused on the special case where $u_n(s) = n^{-s}$. If we multiply two Dirichlet series, this gives us coefficients of the form

$$m^{-s} \cdot n^{-s} = (m \cdot n)^{-s},$$

which is why this is considered to be the 'multiplicative' side of Number Theory and is heavily concerned with prime numbers.
In the case where we have generating functions arising from $u_n(s) = x^n$, multiplying two such series gives us coefficients of the form

$$x^m \cdot x^n = x^{m+n},$$

which is why such generating functions defined by power series are important in the 'additive' side of Number Theory.

14 Generating Functions of the type of 'Lambert Series'

Another type of generating functions is given by the so-called Lambert Series. Inserting $u_n(x) = \frac{x^n}{1-x^n}$ into the definition of a generating series gives us the Lambert Series.

Definition 14.1. The generating function of α_n using the Lambert Series is given by

$$F(x) = \sum_{n=1}^{\infty} \alpha_n \frac{x^n}{1-x^n}.$$

We will not worry about convergence here. Instead we will see such series as formal sums.

Theorem 14.2. *For two generating functions in the form of Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, the following equivalence holds:*

$$\zeta(s)f(s) = g(s) \quad \Leftrightarrow \quad F(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} b_n x^n$$

where $b_n = \sum_{d|n} a_d$.

Proof. We can write

$$\frac{1}{1-x^n} = 1 + (x^n)^1 + (x^n)^2 + (x^n)^3 + ... = \sum_{m=0}^{\infty} x^{mn}.$$

This allows us to write $F(x)$ differently:

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} a_n x^n \left(\sum_{m=0}^{\infty} x^{mn} \right) \\ &= \sum_{n=1}^{\infty} a_n \left(\sum_{m=1}^{\infty} x^{mn} \right) = \sum_{N=1}^{\infty} \left(\sum_{n|N} a_n \right) x^N \end{aligned}$$

Similarly to what we have seen in Section 9, this relation between a_n and b_n is equivalent to

$$\zeta(s)f(s) = g(s).$$

Using the arguments in Section 9, the equivalence in the theorem follows. □

With the use of Theorem 14.2 and some equalities which we have shown in previous sections, we can prove the following two theorems.

Theorem 14.3. *The generating function of the Möbius function $\mu(n)$ using the Lambert Series is given by*

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = x.$$

Proof. We use Theorem 14.2 for $f(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ and $g(s) = 1$. From Theorem 8.2 we know that $\frac{1}{\zeta(s)} = f(s)$. Thus we can see quickly that the following holds

$$f(s) = \frac{1}{\zeta(s)} = \frac{g(s)}{\zeta(s)} \Rightarrow \zeta(s)f(s) = g(s).$$

By Theorem 14.2 we get that

$$F(x) = \sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} b_n x^n = b_1 x^1 + 0 = x,$$

which gives us the desired equality. □

Recall that $\phi(n)$ denotes the number of positive integers smaller than n and which are also coprime to n.

Theorem 14.4. *The generating function of $\phi(n)$ using the Lambert Series is given by*

$$\sum_{n=1}^{\infty} \phi(n) \frac{x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

Proof. Like in the previous proof, we choose specific functions to which Theorem 14.2 can be applied. Here we choose

$$f(s) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \text{ and } g(s) = \zeta(s-1) = \sum_{n=1}^{\infty} \frac{n}{n^s}.$$

In the first part of this presentation we have found the generating function of $\phi(n)$ (Theorem 8.3). Using this, we deduce

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \Leftrightarrow \frac{g(s)}{\zeta(s)} = f(s) \Leftrightarrow \zeta(s)f(s) = g(s).$$

Hence by Theorem 14.2, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty} \phi(n) \frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty} b_n x^n = \sum_{n=1}^{\infty} n x^n = x \left(\sum_{n=1}^{\infty} n x^{n-1} \right) \\ &= x \left(1 + 2x^1 + 3x^2 + 4x^3 + ...\right) = x \left(1 + x + x^2 + x^3 + ...\right)^2 \\ &= \frac{x}{(1-x)^2}.\end{aligned}$$

□

References

[HW] G.H. Hardy and E.M. Wright (1980) *An introduction to the theory of numbers (5th edition)*, Oxford University Press