# Partitions

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## 1 Introduction

A basic problem in additive number theory is to understand in how many ways a positive integer $n$ can be expressed as the sum of integers taken from a set $A = \{a_1, a_2, \ldots\}$.

**Definition 1.** A representation of $n$ as the sum of $a_i \in A$ as above is called a *partition of* $n$.
We denote by $A(n)$ the function that counts the number of partitions of $n$ in $A$.

We will start by giving a few examples to illustrate the notion of partitions.

**Example 1.** (*Representation by Squares*)
We are interested in the number of ways that a positive integer $n$ can be written as the sum of $k$ squares.
For $k \geq 2$, let $r_k(n)$ be the number of solutions to

$$n = x_1^2 + x_2^2 + \ldots + x_k^2,$$

where $x_i \in \mathbb{Z}$ and the order of the summands matters.
Exact expressions of $r_k(n)$ are known for $k = 2, 3, 4, 5, 6, 7, 8$. Also, $r_k(n)$ can be equated with the coefficients of $x^n$ in the power series expansion of the $k$th power of

$$\vartheta = 1 + 2\sum_{n=1}^{\infty} x^{n^2},$$

as noted by Mordell in 1917.

**Example 2.** (*Waring's Problem*)
This can be seen as a generalisation of the above problem. Let $k, s \in \mathbb{Z}_{>0}$ and define $A_k(n)$ to be the number of solutions to the equation

$$n = x_1^k + x_2^k + \ldots + x_s^k.$$

We are interested in determining whether there exists an $s$ depending only on $k$ such that $A_k(n) \geq 1$ for every $n$, i.e., whether the above equation has a solution for every $n$.
If such an $s$ exists, define $g(k)$ to be the minimum value of all such $s$, that is, $g(k) := \min\{s : A_k(n) \geq 1 \ \forall n\}$. In 1909 Hilbert proved the existence of $g(k)$ for all values of $k$ using an inductive argument. Today, all values of $g(k)$ are known except for $k = 4$.

**Example 3.** (*Goldbach Conjecture*)
Define $A(n)$ to be the number of solutions to

$$n = p_1 + p_2,$$

where $p_1, p_2$ are odd primes. In 1742, Goldbach conjectured that $A(n) \geq 1$ for every even $n > 4$, i.e., that every even integer $n > 4$ can be written as the sum of two odd primes. This problem is still undecided today.

### 1.1 Unrestricted Partitions

Let us now consider the set $A = \mathbb{Z}_{>0}$ of all positive integers. Let $A(n)$ be the number of solutions to

$$n = a_{i_1} + a_{i_2} + \dots,$$

where the number of summands is unrestricted, repetitions are allowed and the order of summands is not important.

**Definition 2.** The partition function described above is called the *unrestricted partition function*, and it is denoted by $p(n)$.

**Examples.**

- For $n = 4$ we have $p(4) = 5$, since 4 can be written as

$$\begin{aligned} 4 &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1. \end{aligned}$$

- Similarly, we get $p(5) = 7$:

$$\begin{aligned} 5 &= 4 + 1 \\ &= 3 + 2 \\ &= 3 + 1 + 1 \\ &= 2 + 2 + 1 \\ &= 2 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1. \end{aligned}$$

## 2 Generating Functions

We have already seen generating functions of the form $F(s) = \sum_n f(n) n^{-s}$ defined by Dirichlet series. These are useful in multiplicative number theory, because of the property $n^{-s}m^{-s} = (nm)^{-s}$. In additive number theory, generating functions based on power series

$$F(x) = \sum_n f(n) x^n$$

are better suited, since $x^n x^m = x^{n+m}$.

The following theorem by Euler gives a generating function for the partition function $p(n)$.

**Theorem 1** (Euler)**.** *For $|x| < 1$ we have*

$$\prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n)x^n, \tag{1}$$

*where $p(0) = 1$.*

*Proof.* First, we will give an intuition for the derivation of (1). The formal aspects, including questions of convergence, will be treated more rigorously later.

Begin by noticing that each factor of the product on the left-hand-side can be written as a geometric series

$$\begin{aligned} \frac{1}{1-x^m} &= \sum_{k=0}^{\infty} (x^m)^k \\ &= 1 + \sum_{k=1}^{\infty} x^{mk}, \end{aligned}$$

since we have $|x| < 1$. Now let us treat these as formal power series in one variable, and multiply them together as such. This gives us

$$\begin{aligned} \prod_{m=1}^{\infty} \frac{1}{1-x^m} &= (1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots)(1 + x^3 + x^6 + x^9 + \ldots)\cdots \quad (2) \\ &= 1 + \sum_{k=1}^{\infty} a(k)x^k, \end{aligned}$$

for some coefficients $a(k) \in \mathbb{N}$. We want to argue that $a(k) = p(k)$.

Choose one term from each of the first $m$ factors of the product in (2). Say we take $x^{k_1}$ from the first factor, $x^{2k_2}$ from the second, ..., and $x^{mk_m}$ from the $m$th factor, with $k_i \geq 0$. The product of these $m$ terms is

$$x^{k_1} x^{2k_2} x^{3k_3} \ldots x^{mk_m} = x^k,$$

where $k := k_1 + 2k_2 + 3k_3 + \ldots + mk_m$. This can be written as

$$k = (1 + 1 + \ldots + 1) + (2 + 2 + \ldots + 2) + \ldots + (m + m + \ldots + m),$$

where 1 appears $k_1$ times, 2 appears $k_2$ times and so on. This gives us a partition of $k$.

One can easily see that each partition of $k$ corresponds to exactly one of the terms $x^k$. It immediately follows that $a(k) = p(k)$.

We will now prove the theorem more rigorously. Let us first restrict to the case $0 \leq x < 1$. We can later extend the result using analytic continuation to the disk $|x| < 1$.
Define two functions

$$F_m(x) = \prod_{k=1}^{m} \frac{1}{1-x^k},$$

$$F(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \lim_{m\to\infty} F_m(x).$$

The geometric series $\sum_{k=1}^{\infty} x^k$ clearly converges absolutely. It follows that the product $\prod_{k=1}^{\infty}(1-x^k)$ converges absolutely to a non-zero value, and hence also its reciprocal $\prod_{k=1}^{\infty} \frac{1}{1-x^k} = F(x)$ converges.

Moreover, one can quickly see that the sequence of functions $\{F_m(x)\}$ is pointwise increasing, since

$$F_{m+1}(x) = \frac{1}{1-x^{m+1}} F_m(x) \geq F_m(x).$$

In particular, we get that, for every $m$ and fixed $x$,

$$F(x) \geq F_m(x).$$

Now consider the function $p_m(k)$ which denotes the number of partitions of $k$ into parts which are at most $m$, i.e., the number of solutions to $k = k_1 + 2k_2 + 3k_3 + \ldots + mk_m$. Using the notation from the beginning, $p_m(k) = A_m(k)$, where $A_m = \{1, 2, \ldots, m\}$.

Clearly, we have that

$$p_m(k) \leq p(k) \tag{3}$$

for all $m$, with equality if $m \geq k$.

This tells us that $\lim_{m \to \infty} p_m(k) = p(k)$.

Since $F_m(x)$ clearly converges absolutely (it is a finite product), we can write it as a sum using the $p_m(k)$'s:

$$F_m(x) = 1 + \sum_{k=1}^{\infty} p_m(k) x^k.$$

Splitting this sum gives

$$\begin{aligned} F_m(x) &= 1 + \sum_{k=1}^{\infty} p_m(k) x^k \\ &= \sum_{k=0}^{\infty} p_m(k) x^k \\ &= \sum_{k=0}^{m} p_m(k) x^k + \sum_{k=m+1}^{\infty} p_m(k) x^k \\ &= \sum_{k=0}^{m} p(k) x^k + \sum_{k=m+1}^{\infty} p_m(k) x^k, \end{aligned}$$

where the last step follows from $p_m(k) = p(k)$ for $m \geq k$.

Since $F_m(x)$ converges, so does $\sum_{k=m+1}^{\infty} p_m(k) x^k$, which gives

$$\sum_{k=0}^{m} p(k) x^k \leq F_m(x).$$

Using $F_m(x) \le F(x)$, we get

$$\sum_{k=0}^{m} p(k)x^k \le F_m(x) \le F(x),$$

and hence that $\sum_{k=0}^{\infty} p(k)x^k$ converges. Now we can use (3), i.e. that $p_m(k) \le p(k)$ to get

$$\sum_{k=0}^{\infty} p_m(k)x^k \le \sum_{k=0}^{\infty} p(k)x^k \le F(x).$$

Therefore, $\sum_{k=0}^{\infty} p_m(k)x^k$ converges uniformly in $m$. This allows us to write down the following equalities:

$$\begin{aligned} F(x) &= \lim_{m\to\infty} F_m(x) \\ &= \lim_{m\to\infty} \sum_{k=0}^{\infty} p_m(k)x^k \\ &= \sum_{k=0}^{\infty} \lim_{m\to\infty} p_m(k)x^k \\ &= \sum_{k=0}^{\infty} p(k)x^k. \end{aligned}$$

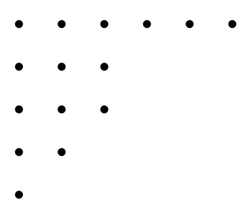
This completes the proof. □

# 3 The Pentagonal Number Theorem

## 3.1 Geometric Representations

There is a natural way of representing partitions geometrically, using a lattice of points. We call this representation the *graph* of a partition.

**Example 4.** The partition $15 = 6 + 3 + 3 + 2 + 1$ is represented as



If we read the graph of a given partition of $n$ vertically, we get a second partition of $n$. Two partitions related in such a way are called *conjugate*. In the example above, the conjugate partition is $15 = 5 + 4 + 3 + 1 + 1 + 1$.

Using this geometric representation, and reading the graphs of a given partition both horizontally and vertically, immediately gives the following two theorems.

**Theorem 2.** *The number of partitions of $n$ into $k$ parts is equal to the number of partitions of $n$ into parts the largest of which is $k$.*

**Theorem 3.** *The number of partitions of $n$ into at most $k$ parts is equal to the number of partitions of $n$ into parts which are at most $k$.*

We will now introduce some terminology that will be useful later on.

**Definition 3.** A graph is said to be in *standard form* if its parts, that is, its rows, are arranged in a strictly decreasing order.

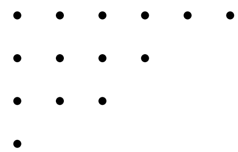


Figure 1: Standard Form

**Definition 4.** The longest line segment connecting the points in the last row of a graph is called the *base*, as illustrated in Figure 2.

Use $b$ to denote the number of points on that line segment, i.e., the number of points in the last row of the graph.

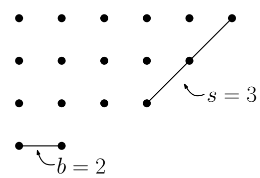



Figure 2: Base and Slope

**Definition 5.** The longest 45° line segment joining the last point of the first row with other points which are the last in their row, is called the *slope* of the graph, as shown in Figure 2.

Use $s$ to denote the number of points on the slope.

Note that both $b \geq 1$ and $s \geq 1$.

### 3.2 Pentagonal Numbers

Consider the arithmetic progression

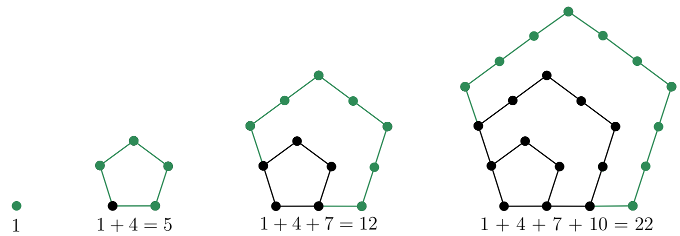
$$1,\ 4,\ 7,\ 10,\ 13, \ldots,\ 3n+1, \ldots$$

and look at its partial sums. Denote by $\omega(n)$ the sum of the first $n$ terms

$$\omega(n) = \sum_{k=0}^{n-1}(3k+1) = \frac{3n(n-1)}{2} + n = \frac{3n^2 - n}{2}.$$

**Definition 6.** The numbers $\omega(n) = \frac{3n^2-n}{2}$ and $\omega(-n) = \frac{3n^2+n}{2}$ are called *pentagonal numbers.*

The first pentagonal numbers are 1, 5, 12, 22, 35, .... They can be represented geometrically in the following way, which justifies their name:




### 3.3 Euler's Pentagonal Number Theorem

We now consider the function $\prod_{m=1}^{\infty}(1-x^m)$. This is the reciprocal of the generating function of $p(n)$ seen earlier, which guarantees convergence. Write

$$\prod_{m=1}^{\infty}(1-x^m) = 1 + \sum_{n=1}^{\infty} a(n)x^n, \tag{4}$$

where we claim that $a(n)$ is a partition function, which we want to determine.

The product on the left of (4) produces terms of the form $\pm x^n$, where the coefficient is $+1$ if $x^n$ is the product of an even number of terms, and $-1$ if it is the product of an odd number of terms.

Denote by $p_e(n)$ the number of partitions of $n$ into an even number of unequal parts, and likewise by $p_o(n)$ the number of partitions of $n$ into an odd number of unequal parts. This gives us

$$a(n) = p_e(n) - p_o(n)$$

Euler showed that $p_e(n) = p_o(n)$ for all $n$ except the pentagonal numbers.

**Theorem 4** (Pentagonal Number Theorem)**.** *If* $|x| < 1$*, we have*

$$\begin{aligned}\prod_{m=1}^{\infty}(1-x^m) &= 1 + \sum_{n=1}^{\infty}(-1)^n(x^{\omega(n)} + x^{\omega(-n)}) \\ &= \sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)}.\end{aligned}$$

This theorem was proven by Euler in 1750, using induction. The following proof, due to Franklin in 1881, is a combinatorial one, and makes use of the graphical representation of partitions.

*Proof.* We already saw that

$$\prod_{m=1}^{\infty}(1-x^m) = \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n.$$

We want to show that there exists a one-to-one correspondence between partitions of $n$ with an even number of unequal parts and partitions with an odd number of unequal parts, except when $n$ is a pentagonal number.

Consider the graphs of all partitions of $n$, in standard form. The latter guarantees that each partition is associated to a unique graph. Define two operations $A$ and $B$ on the graph of a partition:

- $A$: move the points on the base in such a way that they are parallel to the slope of the graph, and one endpoint of the base is now in the top row. In other words, the base of the graph becomes the new slope.

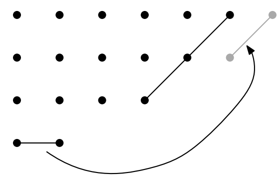


Figure 3: Operation A

- $B$: move the points on the slope in such a way that they are parallel to the base, i.e., they form an additional row beneath the base. In other words, the slope becomes the new base of the graph.

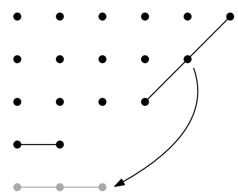


Figure 4: Operation B

We call an operation *permissible* if it preserves the standard form of the graph.

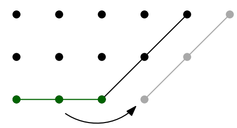
Note that, for a given graph, if operation $A$ is permissible, then it produces a partition with exactly one part less. Similarly, if $B$ is permissible, it produces a partition with one part more.

Fix a natural number $n$; if, for every partition, either operation $A$ or operation $B$ is permissible but not both, then there is a one-to-one correspondence between the partitions of $n$ into odd and even unequal parts, i.e., $p_e(n) = p_o(n)$. To see for which values of $n$ this holds, we consider three different cases.

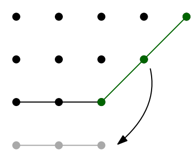
*Case 1*: $b < s$
If the size of the base is strictly smaller than the slope, then operation $A$ is always permissible. Operation $B$ is never permissible, since the number of points that is moved below the base is larger than the number of points in the base, so the standard form is not preserved.

*Case 2*: $b = s$

Operation $A$ is permissible, unless the slope and the base intersect. In that case, $s$ is equal to the number of rows in the graph, i.e., the number of parts of the partition. Since $b = s$, and as moving the base decreases the number of rows, the base in its new position will be too long for the rest of the graph, as seen in the diagram below.



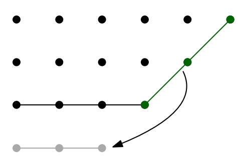
Operation $B$, on the other hand, is never permissible. As shown below, the last row of the resulting graph will be longer by one point than the row above.



*Case 3*: $b > s$

Operation $A$ is never permissible, by a similar argument as in Case 2.

Operation $B$ is permissible, unless $b = s + 1$ and the slope and base intersect. In that case, the resulting graph will have two parts of equal length, as can be seen in the diagram below.



To summarize, for every $n$ we have that either operation $A$ or operation $B$ is permissible, except for one of the following two exceptions:

(i) $b = s$ and base and slope intersect;

(ii) $b = s + 1$ and base and slope intersect.

Consider a partition of a number $n$ whose graph is of the form described in $(i)$. Then the number of rows, or parts, of the graph is equal to $s$. Since $s = b$, the slope contains the last point from every row. In other words, each part, when starting at the bottom of the graph, increases by exactly 1. Lets say the partition has $k$ parts, so $k = s = b$. Then

$$\begin{aligned} n &= k + (k+1) + (k+2) + \ldots + (2k-1) \\ &= \frac{3k^2 - k}{2} = \omega(k). \end{aligned}$$

If $k$ is even, then there is an extra partition of $n$ into even unequal parts, since neither operation $A$ nor $B$ can be used to produce a corresponding partition into odd unequal parts. Similarly, if $k$ is odd, we have an extra partition into odd unequal parts. It follows that

$$p_e(n) - p_o(n) = (-1)^k.$$

Finally, consider a partition whose graph is of the form described in $(ii)$. Again, denote by $k$ the number of parts. We have that $s = k$, and therefore $b = k + 1$, which means that each row has one extra point when compared to the case above. So we have

$$\begin{aligned} n &= \frac{3k^2 - k}{2} + k \\ &= \frac{3k^2 + k}{2} = \omega(-k). \end{aligned}$$

As above, one can see that again

$$p_e(n) - p_o(n) = (-1)^k.$$

This completes the proof. □

## 4 Euler's Recursion Formula

Using the results we have seen so far, one can easily derive a recursive formula for the partition function $p(n)$.

**Theorem 5** (Euler)**.** *For $n \geq 1$ we have*

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \ldots = 0,$$

*or equivalently,*

$$p(n) = \sum_{k=1}^{\infty} (-1)^{k+1} (p(n - \omega(k)) + p(n - \omega(-k))).$$

*Proof.* Recall from Theorem 1,

$$\prod_{m=1}^{\infty} \frac{1}{1 - x^m} = \sum_{n=0}^{\infty} p(n) x^n,$$

and from the Pentagonal Number Theorem (Theorem 4)

$$\prod_{m=1}^{\infty} (1 - x^m) = 1 + \sum_{n=1}^{\infty} (-1)^n (x^{\omega(n)} + x^{\omega(-n)}).$$

Multiplying these two equations together gives

$$\left(1 + \sum_{n=1}^{\infty} (-1)^n (x^{\omega(n)} + x^{\omega(-n)})\right) \left(\sum_{n=0}^{\infty} p(n) x^n\right) = 1.$$

We can now equate coefficients of $x^n$ to get the result. □

## 5 Logarithmic differentiation

In this section, we want to obtain another recursion formula for $p(n)$. For this, we will use a technique called logarithmic differentiation. This technique makes use of the fact that, for a differentiable function $f$, it holds

$$(\log f)' = \frac{f'}{f}\,.$$

We also call the fraction $\frac{f'}{f}$ the *logarithmic derivative* of $f$. Logarithmic differentiation is often used when dealing with products, since the logarithm turns products into sums, which are easier to differentiate ([3]).

The following theorem will give us the desired recursion formula for $p(n)$ as a special case.

**Theorem 6.** *Let $A$ be a given set of positive integers and $f(n)$ an arithmetical function such that the functions*

$$F_A(x) := \prod_{n\in A}(1-x^n)^{-\frac{f(n)}{n}}\,, \quad G_A(x) := \sum_{n\in A}\frac{f(n)}{n}x^n$$

*converge absolutely for $|x|<1$ and are analytic in the unit disk $|x|<1$.*

*Then the numbers $p_{A,f}(n)$ defined by the equation*

$$F_A(x) = \prod_{n\in A}(1-x^n)^{-\frac{f(n)}{n}} = 1+\sum_{n=1}^{\infty}p_{A,f}(n)x^n$$

*satisfy the recursion formula*

$$np_{A,f}(n) = \sum_{k=1}^{n} f_A(k)p_{A,f}(n-k)\,,$$

*where $p_{A,f}(0)=1$ and*

$$f_A(k) = \sum_{\substack{d|k\\ d\in A}} f(d)\,.$$

*Proof.* We want to use logarithmic differentiation, so we first compute

$$\log F_A(x) = -\sum_{n\in A}\frac{f(n)}{n}\log(1-x^n)\,.$$

Using the Taylor expansion $\log(1-z) = -\sum\frac{z^k}{k}$, we find

$$\begin{aligned}\log F_A(x) &= \sum_{n\in A}\frac{f(n)}{n}\sum_{m=1}^{\infty}\frac{x^{nm}}{m}\\ &= \sum_{m=1}^{\infty}\frac{1}{m}G_A(x^m)\,.\end{aligned}$$

Using logarithmic differentiation, we get

$$\begin{aligned}
x\frac{F_A'(x)}{F_A(x)} &= x\left(\log F_A(x)\right)' \\
&= x\left(\sum_{m=1}^{\infty}\frac{1}{m}G_A(x^m)\right)' \\
&= \sum_{m=1}^{\infty} x^m G_A'(x^m) \\
&= \sum_{m=1}^{\infty} x^m \sum_{n\in A} f(n)x^{m(n-1)} \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \chi_A(n)f(n)x^{mn}\,,
\end{aligned}$$

where $\chi_A$ is the characteristic function of $A$. Now, we combine all terms with exponent $k = mn$. Using the definition of $f_A(k)$, we find

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \chi_A(n)f(n)x^{mn} &= \sum_{k=1}^{\infty}\sum_{d|k}\chi_A(d)f(d)x^k \\
&= \sum_{k=1}^{\infty} f_A(k)x^k\,.
\end{aligned}$$

Thus we have

$$xF_A'(x) = F_A(x)\sum_{k=1}^{\infty} f_A(k)x^k\,. \tag{5}$$

By definition of $p_{A,f}$, we can also write

$$F_A(x) = \sum_{n=0}^{\infty} p_{A,f}(n)x^n\,.$$

Hence

$$xF_A'(x) = \sum_{n=0}^{\infty} np_{A,f}(n)x^n\,,$$

and

$$\begin{aligned}
F_A(x)\sum_{k=1}^{\infty} f_A(k)x^k &= \sum_{n=0}^{\infty} p_{A,f}(n)x^n \sum_{k=1}^{\infty} f_A(k)x^k \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{n} p_{A,f}(n-k)f_A(k)x^n\,.
\end{aligned}$$

Thus, by (5), we have

$$\sum_{n=0}^{\infty} np_{A,f}(n)x^n = \sum_{n=0}^{\infty}\sum_{k=1}^{n} p_{A,f}(n-k)f_A(k)x^n\,.$$

By comparing coefficients, we get the desired recursion formula:

$$np_{A,f}(n) = \sum_{k=1}^{n} p_{A,f}(n-k) f_A(k)\,.$$

□

Now we look at two applications of this theorem. First, we derive another recursion formula for $p(n)$.

**Example 5.** We take $A = \mathbb{N}$ and $f(n) = n$. Then, by Euler's Theorem (Theorem 1), it follows that $p_{A,f}(n) = p(n)$. Thus, by Theorem 6 we have the recursion formula

$$np(n) = \sum_{k=1}^{n} \sigma(n) p(n-k)\,,$$

where $\sigma(k) = \sum_{d|k} d$ is the sum over all divisors of $k$.

This result is quite remarkable, since it links the function $\sigma$ of multiplicative number theory to the function $p$ of additive number theory.

Next, we find a recursion formula for $\sigma(k)$ using Euler's Pentagonal Number Theorem.

**Example 6.** We take $A = \mathbb{N}$ and $f(n) = -n$. Then, by Euler's Pentagonal Number Theorem (Theorem 4), we have that

$$p_{A,f}(n) = \begin{cases} (-1)^m & \text{if } n = \omega(m) \text{ or } n = \omega(-m)\,, \\ 0 & \text{if } n \text{ is not a pentagonal number}\,. \end{cases} \tag{6}$$

Theorem 6 gives us

$$np_{A,f}(n) = -\sum_{k=1}^{n} \sigma(k) p_{A,f}(n-k) = -\sum_{k=0}^{n-1} \sigma(n-k) p_{A,f}(k)\,.$$

Using (6), we can rewrite this as

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \cdots = \begin{cases} (-1)^{m-1}\omega(m) & \text{if } n = \omega(m)\,, \\ (-1)^{m-1}\omega(-m) & \text{if } n = \omega(-m)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where the sum on the right hand side terminates for $k \leq 1$ in $\sigma(k)$. So we have,

$$\begin{aligned} &\sigma(1) = 1 \\ &\sigma(2) = \sigma(1) + 2 = 3 \\ &\sigma(3) = \sigma(2) + \sigma(1) = 4 \\ &\sigma(4) = \sigma(3) + \sigma(2) = 7 \\ &\sigma(5) = \sigma(4) + \sigma(3) - 5 = 6 \\ &\ldots \end{aligned}$$

## 6 Jacobi's Triple Product Identity

In this section we prove the celebrated Jacobi's Triple Product Identity. This formula gives many partition identities as special cases, for example Euler's Pentagonal Number Theorem. [1, p. 318]

**Theorem 7** (Jacobi's Triple Product Identity)**.** *For complex $x$ and $z$ with $|x| < 1$ and $z \neq 0$ we have*

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m} . \tag{7}$$

Before we prove this statement, we first derive Euler's Pentagonal Number Theorem (Theorem 4) from it.

**Example 7.** We replace $x$ with $x^a$ and $z^2$ with $-x^b$ in (7) to obtain

$$\prod_{n=1}^{\infty}(1-x^{2an})(1-x^{2na-a+b})(1-x^{2na-a-b}) = \sum_{m=-\infty}^{\infty} (-1)^m x^{am^2+bm} .$$

By setting $a = \frac{3}{2}$ and $b = \frac{1}{2}$, we get Euler's Pentagonal Number Theorem:

$$\begin{aligned} \prod_{n=1}^{\infty}(1-x^n) &= \prod_{n=1}^{\infty}(1-x^{3n})(1-x^{3n-1})(1-x^{3n-2}) \\ &= \sum_{m=-\infty}^{\infty} (-1)^m x^{\frac{3m^2+m}{2}} \\ &= \sum_{m=-\infty}^{\infty} (-1)^m x^{\omega(m)} . \end{aligned}$$

*Proof of Theorem 7.* First, note that for $|x| < 1$ and $z \neq 0$ the products

$$\prod_{n=1}^{\infty}(1-x^{2n}) , \quad \prod_{n=1}^{\infty}(1+x^{2n-1}z^2) , \quad \prod_{n=1}^{\infty}(1+x^{2n-1}z^{-2}) \tag{8}$$

all converge absolutely, since the series

$$\sum_{n=1}^{\infty} x^{2n} , \quad \sum_{n=1}^{\infty} x^{2n-1}z^2 , \quad \sum_{n=1}^{\infty} x^{2n-1}z^{-2} \tag{9}$$

converge absolutely. The series

$$\sum_{m=-\infty}^{\infty} x^{m^2} z^{2m} . \tag{10}$$

also converges absolutely, for example by the root test.

Further, we fix $x$ and restrict $z$ to a compact subset $K$ of $\mathbb{C} \setminus \{0\}$. In particular, $z$ is bounded by a constant $C$. Since the series

$$\sum_{m=-\infty}^{\infty} |x|^{m^2} C^{2m}$$

converges by the root test, we can apply the Weierstrass M-Test and find that the series (10) converges uniformly on $K$. Since $K$ was an arbitrary compact subset, we conclude that the series (10) is an analytic function of $z$ for $z \neq 0$. (See Theorem 10.28 in [2, p. 214])

If we instead fix $z \neq 0$ and restrict $x$ to the disk $|x| \leq r < 1$, we have that the series

$$\sum_{m=-\infty}^{\infty} r^{m^2} |z|^{2m}$$

converges by the root test. Hence, by the Weierstrass M-Test, the series (10) converges absolutely on the disk $|x| \leq r$ and thus the series is an analytic function of $x$ in the disk $|x| < 1$.

Applying the same arguments to the series (9), we conclude that the products (8) are analytic functions of $z$ in $\mathbb{C} \setminus \{0\}$ and analytic functions of $x$ in the disk $|x| < 0$.

Now, let us fix an $x$ with $|x| < 1$ and define, for $z \neq 0$,

$$F(z) := \prod_{n=1}^{\infty} (1 + x^{2n-1} z^2)(1 + x^{2n-1} z^{-2}).$$

We claim that, for $x \neq 0$

$$xz^2 F(xz) = F(z). \tag{11}$$

Using the definition of $F$, we find

$$\begin{aligned} F(xz) &= \prod_{n=1}^{\infty} (1 + x^{2n+1} z^2)(1 + x^{2n-3} z^{-2}) \\ &= \prod_{n=2}^{\infty} (1 + x^{2n-1} z^2) \prod_{m=0}^{\infty} (1 + x^{2m-1} z^{-2}) \\ &= \frac{(1 + x^{-1} z^{-2})}{(1 + xz^2)} F(z). \end{aligned}$$

Since $xz^2(1 + x^{-1}z^{-2}) = (1 + xz^2)$, we have shown our claim.

Now we define $G$ as the left hand side of (7):

$$G(z) := F(z) \prod_{n=1}^{\infty} (1 - x^{2n}).$$

Since the infinite product doesn't depend on $z$, $G$ also satisfies the functional equation (11) for $x \neq 0$. Furthermore, $G$ is an even function of $z$, and, as a uniformly converging product of analytic functions, analytic for $z \neq 0$. Thus there exists a Laurent expansion of $G$ around $z = 0$ of the form

$$G(z) = \sum_{m=-\infty}^{\infty} a_m z^{2m}, \tag{12}$$

where the coefficients $a_m$ depend on $x$. Since $G$ is even, the Laurent coefficients for the odd powers of $z$ cancel, and we are left with only the terms of even power. Further, since $G(z) = G(z^{-1})$, it holds $a_m = a_{-m}$.

Using equations (11) and (12), we find, for $x \neq 0$,

$$\sum_{m=-\infty}^{\infty} a_m z^{2m} = xz^2 \sum_{m=-\infty}^{\infty} a_m x^{2m} z^{2m}$$
$$= \sum_{m=-\infty}^{\infty} a_m x^{2m+1} z^{2(m-1)}$$
$$= \sum_{m=-\infty}^{\infty} a_{m-1} x^{2m-1} z^{2m} .$$

By comparing coefficients, we obtain the following recursion formula for $a_m$:

$$a_m = a_{m-1} x^{2m-1} , \quad \text{for } x \neq 0 .$$

Iterating this recursion, we get

$$a_m = a_0 x^{1+3+\cdots+2m-1} = a_0 x^{m^2} , \quad \text{for } x \neq 0 ,$$

where we used $\sum_{k=1}^{m} 2k - 1 = m^2$ in the last equality.

From now on, let us write $G_x(z)$ and $a_0(x)$ to denote the dependence on $x$.

For $x = 0$, we have by definition of $G$ that $G_0(z) \equiv 1$. Hence

$$a_m(0) = \begin{cases} 0 & m \neq 0 , \\ 1 & m = 0 , \end{cases}$$

Thus, we have reached

$$G_x(z) = a_0(x) \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m} , \quad \text{for } |x| < 1 . \tag{13}$$

It remains to show that $a_0(x) = 1$ for all $x$ with $|x| < 1$.

For this, we first note that $a_0(x) \to 1$ as $x \to 0$. since $G_0(z) = 1$.

Setting $z = e^{\frac{\pi i}{4}}$ in (13), we find

$$\frac{G_x(e^{\frac{\pi i}{4}})}{a_0(x)} = \sum_{m=-\infty}^{\infty} x^{m^2} e^{\frac{\pi i m}{2}} = \sum_{m=-\infty}^{\infty} x^{m^2} i^m .$$

If $m$ is odd, we have $i^m = -i^{-m}$, and thus the terms for odd $m$ cancel. For $m = 2n$, we have $i^m = (-1)^n$. Thus we are left with

$$\frac{G_x(e^{\frac{\pi i}{4}})}{a_0(x)} = \sum_{n=-\infty}^{\infty} (-1)^n x^{(2n)^2} = \frac{G_{x^4}(i)}{a_0(x^4)} ,$$

where we used (13) again in the last equality.

Next, we want to show that $G_x(e^{\frac{\pi i}{4}}) = G_{x^4}(i)$. To this end, we use the definitions of $F$ and $G$:

$$\begin{aligned} G_x(e^{\frac{\pi i}{4}}) &= \prod_{n=1}^{\infty}(1-x^{2n})(1+ix^{2n-1})(1-ix^{2n-1}) \\ &= \prod_{n=1}^{\infty}(1-x^{2n})(1+x^{4n-2}). \end{aligned}$$

Now, we can write the even exponents $2n$ as $4m$ if $n$ is even, or as $4m-2$ if $n$ is odd. We get

$$\begin{aligned} G_x(e^{\frac{\pi i}{4}}) &= \prod_{n=1}^{\infty}(1-x^{4n})(1-x^{4n-2})(1+x^{4n-2}) \\ &= \prod_{n=1}^{\infty}(1-x^{4n})(1-x^{8n-4}). \end{aligned}$$

We apply the same trick as before for the exponents $4n$ and conclude

$$\begin{aligned} G_x(e^{\frac{\pi i}{4}}) &= \prod_{n=1}^{\infty}(1-x^{8n})(1-x^{8n-4})(1-x^{8n-4}) \\ &= G_{x^4}(i), \end{aligned}$$

where we used the definition of $G$ and $F$ in the last equality.

Thus, we deduce that $a_0(x) = a_0(x^4)$. By iterating this equality, we find

$$a_0(x) = a_0(x^{4k}) \quad k = 1, 2, \ldots$$

Since $|x| < 1$, we have $x^{4k} \to 0$ as $k \to \infty$, and therefore

$$a_0(x) = \lim_{k\to\infty} a_0(x^k) = \lim_{x\to 0} a_0(x) = 1.$$

□

# 7 Growth of $p(n)$

In this last section, we have a quick look at the asymptotic growth rate of $p(n)$. The number of unrestricted partitions $p(n)$ grows very quickly with $n$. An asymptotic formula of Rademacher states that

$$p(n) \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}} \quad \text{as } n \to \infty \quad \text{for } K = \pi\left(\frac{2}{3}\right)^{\frac{1}{2}}$$

For more details, see [1, p. 316]. In the following we will state and prove a much simpler upper bound for $p(n)$.

**Theorem 8.** *If $n \geq 1$, we have*

$$p(n) < e^{K\sqrt{n}},$$

*where $K = \pi\left(\frac{2}{3}\right)^{\frac{1}{2}}$.*

*Proof.* First, we consider as before the function

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = 1 + \sum_{k=1}^{\infty} p(k) x^k$$

for $0 < x < 1$. Clearly $p(n)x^n < F(x)$. By taking the logarithm of this identity and moving $n \log x$ to the other side, we get

$$\log p(n) < \log F(x) + n \log \frac{1}{x} . \tag{14}$$

Next, we estimate $\log F(x)$. By definition of $F$,

$$\begin{aligned} \log F(x) &= -\log \prod_{n=1}^{\infty} (1-x^n) \\ &= -\sum_{n=1}^{\infty} \log(1-x^n) . \end{aligned}$$

Using the Taylor expansion $\log(1-z) = -\sum \frac{z^k}{k}$, we find

$$\begin{aligned} \log F(x) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} x^{mn} \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} (x^m)^n \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{x^m}{1-x^m} , \end{aligned} \tag{15}$$

where we used the geometric series in the last equality.

Now note that

$$\frac{1-x^m}{1-x} = \sum_{k=0}^{m-1} x^k .$$

Since $x < 1$, we have $x^{m-1} < x^k$ for all $k = 0, \ldots, m-2$ and hence

$$m x^{m-1} < \frac{1-x^m}{1-x} .$$

We multiply the above inequality with $\frac{1-x}{x^m}$ to get

$$\frac{m(1-x)}{x} < \frac{1-x^m}{x^m} .$$

Taking the inverses on both sides and dividing by $m$, we get

$$\frac{1}{m} \frac{x^m}{1-x^m} < \frac{1}{m^2} \frac{x}{1-x} .$$

Combining this with (15), we have

$$\begin{aligned}\log F(x) &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{x^m}{1-x^m} \\ &< \frac{x}{1-x} \sum_{m=1}^{\infty} \frac{1}{m^2} \\ &= \frac{x}{1-x} \frac{\pi^2}{6} \\ &= \frac{\pi^2}{6t}\,,\end{aligned}$$

where $t = \frac{1-x}{x}$.

Next, we estimate $\log \frac{1}{x}$. Note that, for $s > 0$, it holds that $\log(1+s) < s$. Since by definition of $t$

$$1 + t = 1 + \frac{1-x}{x} = \frac{1}{x}\,,$$

we have

$$\log \frac{1}{x} = \log(1+t) < t\,.$$

Combining our two estimates with (14), we find

$$\log p(n) < \frac{\pi^2}{6t} + nt \quad \forall t$$

The right hand side attains its minimal value for $t = \frac{\pi}{\sqrt{6n}}$. For this value, we have

$$\log p(n) < \frac{2n\pi}{\sqrt{6n}} = K\sqrt{n}\,,$$

and the theorem follows. □

## References


[1] Tom. M. Apostol. *Introduction to Analytic Number Theory*. Springer, 1976.

[2] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 1987.

[3] Wikipedia contributors. Logarithmic differentiation — Wikipedia, the free encyclopedia, 2021. [Online; accessed 29-November-2021].