Seminar in Elementary Number Theory

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Abstract

This document is the notes of a seminar in elementary number theory on the topic of Diophantine approximations. The main goal is to present two different proofs of a theorem of Hurwitz. We will touch upon the motivations that led to the theorem and then present a proof using continued fractions and a proof by L.R. Ford which relies on geometry [For25].

1 Introduction

Diophantine approximation deals with the approximation of real numbers by rational numbers. In the following, we assume that α is an irrational number and $\frac{p}{q}$ an irreducible fraction. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the quantity $\left|\alpha - \frac{p}{q}\right|$ can be obviously small, therefore we want to estimate the distance in terms of the denominator. The easiest example is that we can find $p, q \in \mathbb{Z}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q}$.

In the following, we will assume that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{p}{q}$ is irreducible.

Theorem 1 (Dirichlet)**.** *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. For all $N \in \mathbb{N}$, there exists $\frac{p}{q} \in \mathbb{Q}, q \leq N$ such that $\left|\xi - \frac{p}{q}\right| < \frac{1}{qN}$.*

Remark It suffices to prove for $0 < \alpha < 1$ since $\alpha - \frac{p}{q} = \alpha - n - \frac{p-qn}{q}$, taking $n = \lfloor \alpha \rfloor$ reduces the problem to the interval $(0,1)$.

Proof. [Cha68] Let $x_n = n\alpha - \lfloor n\alpha \rfloor \in (0,1)$ and divide $(0,1)$ into N intervals $(0, \frac{1}{N}), ..., (\frac{N-1}{N}, 1)$. Then there are two possible cases:

Case 1: $\exists x_k \in (0, \frac{1}{N})$. This implies $0 < k\alpha - |k\alpha| < \frac{1}{N}$. Dividing by k we get $\alpha - |k\alpha| < \frac{1}{kN}$. Let $p = |k\alpha|$ and $q = k$ and we're done.

Case 2: None of the x_k's lies in $(0, \frac{1}{N})$. Then by the pigeonhole principle, there exists an interval that contains two or more x_k's, say x_n and x_m, $n > m$. Hence

$$|x_n - x_m| < \frac{1}{N} \implies \left|\alpha - \frac{\lfloor n\alpha \rfloor - \lfloor m\alpha \rfloor}{n - m}\right| < \frac{1}{(n-m)N}$$

Set $p = \lfloor n\alpha \rfloor - \lfloor m\alpha \rfloor, q = n - m$. ∎

Theorem 2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. For all $N \in \mathbb{N}$, there exists $\frac{p}{q} \in \mathbb{Q}, q \leq N$ such that $\left|\xi - \frac{p}{q}\right| < \frac{1}{q(N+1)}$.*

Proof. The proof is similar to the one of theorem 1, adding $x_0 := 0, x_{N+1} := 1$ so that $\sum_{n=0}^{N} x_{n+1} - x_n = x_{N+1} - x_0 = 1$. Since there are $N+1$ irrational terms, at least one term is strictly less than $\frac{1}{N+1}$, which allows us to conclude as in theorem 1. □

Remark Theorem 1 tells us that for any irrational α, there exists an infinity of rational numbers $\frac{p}{q}$ that fulfill $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$. It is natural to ask ourselves: can we improve this bound? The following theorem (without proof) tells us that we cannot increase the exponent of q.

Theorem 3 (Roth's theorem [DR55]). *Every irrational algebraic number α has approximation exponent equal to 2, i.e. $\forall \epsilon > 0$ the inequality*

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{q^{2+\epsilon}}$$

is satisfied only by finitely many fractions $\frac{p}{q} \in \mathbb{Q}$.

Now the question to ask is: does there exist $\lambda > 1$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\lambda q^2}$ holds for infinitely many irrationals? If so, what would be the biggest such λ? We will prove that this is true for $\lambda = 2$.

Theorem 4. *If α is irrational, there exists infinitely many irreducible fractions $\frac{p}{q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.*

The proof uses Farey sequences.

1.1 Farey Sequences

Definition 1.1 (Farey Sequence) Let $n \geq 1$. By convention, a Farey sequence is a sequence of fractions that starts with $\frac{0}{1}$ and ends with $\frac{1}{1}$. The n-th Farey sequence is composed of all irreducible fractions $\frac{p}{q}$ such that $q \leq N$, in ascending order.

Example

1. $F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$
2. $F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$
3. $F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$

Lemma 1.1. *Let $\frac{a}{b} < \frac{c}{d}$ be two irreducible fractions. Then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ and $\frac{a+c}{b+d}$ is irreducible. $\frac{a+c}{b+d}$ is called the mediant of $\frac{a}{b}$ and $\frac{c}{d}$.*

Theorem 5. *If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in any row, then for all F_n that contain $\frac{a+c}{b+d}$, $\frac{a+c}{b+d}$ has the smallest denominator of any rational number between $\frac{a}{b}$ and $\frac{c}{d}$ and is the unique rational number between $\frac{a}{b}$ and $\frac{c}{d}$ with denominator $b+d$.*

Proof. Let $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$ be irreducible fractions.

$$\frac{c}{d} - \frac{a}{b} = \frac{c}{d} - \frac{x}{y} + \frac{x}{y} - \frac{a}{b} \geq \frac{1}{dy} + \frac{1}{by} = \frac{b+d}{bdy}$$
$$\frac{b+d}{bdy} \leq \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd} = \frac{1}{bd} \implies y \geq b+d$$

If $y = b + d$, then all the inequalities become equalities, thus $cy - dx = bx - ay = 1$ and so $x = a + c, y = b + d$. [Uni] □

Corollary. *In a Farey sequence, each number is the mediant of its two neighbors.*

Theorem 6. *Given $0 \leq \frac{a}{b} \leq \frac{c}{d} \leq 1$, $\frac{a}{b}, \frac{c}{d}$ are Farey neighbors in F_n for some n if and only if $bc - ad = 1$.*

Proof. $\Rightarrow$: By induction on n. It's obvious for F_1. Suppose, then $\frac{a}{b}, \frac{c}{d}$ are Farey neighbors in F_n, then either they are still neighbors in F_{n+1} or $b + d \leq n + 1$ and $\frac{a}{b} < \frac{a+b}{c+d} < \frac{c}{d}$ are consecutive in F_{n+1}. In either case the claim is true.

$\Leftarrow$: Let's consider three rationals $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ with $bp - ap = qc - qd = 1$. Then $bp + pd = qc + aq \implies p(b + d) = q(a + c) \implies \frac{p}{q} = \frac{a+c}{b+d}$, so they are Farey neighbors in F_{a+b}. □

1.2 Proof of Theorem 3

Let $N \geq 1$ and lt $\frac{a}{b} < \alpha < \frac{c}{d}$ be consecutive Farey numbers in F_N. Claim: $\alpha - \frac{a}{b} < \frac{1}{2b^2}$ or $\frac{c}{d} - \alpha < \frac{1}{2d^2}$, so choose $\frac{p}{q}$ to be $\frac{a}{b}$ or $\frac{c}{d}$. Indeed, if not, then

$$\alpha - \frac{a}{b} > \frac{1}{2b^2}, \frac{c}{d} - \alpha > \frac{1}{2d^2}$$

(strict inequalities since $\alpha \notin \mathbb{Q}$). On the one hand,

$$\frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd} = \frac{1}{bd}$$

by a property of the Farey sequence. On the other hand,

$$\frac{c}{d} - \frac{a}{b} > \frac{1}{2d^2} - \frac{1}{2b^2} = \frac{b^2-d^2}{2b^2d^2}$$
$$\implies \frac{1}{bd} - \frac{b^2-d^2}{2b^2d^2} = -\frac{(b-d)^2}{2b^2d^2} < 0$$

Contradiction. Since $N \in \mathbb{N}$ is arbitrary, we are done.

1.3 Hurwitz's Theorem

Now it's time to look at the main result, which was stated and proved by Hurwitz in 1891 [Hur91]. Adolf Hurwitz was a German mathematician who worked on algebra, analysis, geometry and number theory. We will first present his original proof that uses continued fractions, and then we will show a geometric proof published by L.R.Ford in 1916 [For25].

Theorem 7 (Hurwitz)**.**
i) If $0 < \lambda \leq \sqrt{5}$, every irrational number α can be approximated by infinitely many fractions $\frac{p}{q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\lambda q^2}$.
ii) $\sqrt{5}$ is the best bound, i.e. if $\lambda > 5$, there exist irrational numbers α such that $\alpha - \frac{p}{q} < \frac{1}{\lambda q^2}$

In order to prove this theorem, we need to introduce one more important tool: continued fractions. Any irrational number α can be written as a continued fraction.

$$\alpha = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}} := [a_1; a_2, a_3, ...] \qquad a_n \in \mathbb{Z}$$

In the following we will assume $\alpha > 0$ and hence $a_n > 0, \forall n$. If $\alpha < 0$, it suffices to add a negative sign or to take $a_1 = \lfloor \alpha \rfloor$ so that the continued fraction still represents a positive irrational number. Now since the continued fraction part is less than 1, it must be that $a_1 = \lfloor \alpha \rfloor$, and the rest can be determined recursively. Let $\alpha_0 = \alpha$ and define

$$\alpha_0 = a_1 + \frac{1}{\alpha_1}, \quad a_1 = \lfloor \alpha_0 \rfloor, \quad \alpha_1 \in \mathbb{R}$$
$$\alpha_1 = a_1 + \frac{1}{\alpha_2}, \quad a_2 = \lfloor \alpha_1 \rfloor, \quad \alpha_2 \in \mathbb{R}$$
$$...$$
$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}, \quad a_{n+1} = \lfloor \alpha_n \rfloor, \quad \alpha_n \in \mathbb{R}$$
$$...$$

We can write $\alpha = [a_1; a_2, ..., \alpha_n], \forall n \in \mathbb{N}$, and define

$$\frac{p_n}{q_n} := [a_1; ..., a_n], \quad \frac{p_n}{q_n} \xrightarrow{n \to \infty} \alpha$$

$\frac{p_n}{q_n}$ are called convergents since they converge to α. Also notice that $a_n > 0$ by definition and $\alpha_n = [a_{n+1}; a_{n+2}, ...]$.

Now let us look at some useful properties.

Theorem 8.

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} & p_0 := 1,\ p_1 := a_1 \\ q_n = a_n q_{n-1} + q_{n-2} & q_0 := 0,\ q_1 := 1 \end{cases}$$

Proof. The proof uses induction on n:

$n = 1: \frac{p_1}{q_1} = \frac{a_1}{1}$

$n = 2: \frac{p_2}{q_2} = a_1 + \frac{1}{a_2} = \frac{a_2 a_1 + 1}{a_2} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0}$

$n - 1 \to n$: Suppose the formula is true for any sequence $[x_1; ..., x_{n-1}]$ with $n - 1$ terms

$$\frac{p_n}{q_n} := [a_1; ..., a_n] = [\underbrace{a_1}_{\widetilde{a_1}}; ..., \underbrace{a_{n-1} + \frac{1}{a_n}}_{\widetilde{a}_{n-1}}] \overset{hyp.}{=} \frac{\widetilde{a}_{n-1} p_{n-2} + p_{n-3}}{\widetilde{a}_{n-1} q_{n-2} + q_{n-3}} = \frac{(a_n a_{n-1} + 1) p_{n-2} + a_n p_{n-3}}{(a_n a_{n-1} + 1) q_{n-2} + a_n q_{n-3}}$$

$$= \frac{a_n(a_{n-1} p_{n-2} + p_{n-3} + p_{n-2}}{a_n(a_{n-1} q_{n-2} + q_{n-3} + q_{n-2}} = \frac{a_n p_{n-2} + p_{n-2}}{a_n q_{n-2} + q_{n-2}}$$

□

Theorem 9. $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$

Proof. Write the above relations in matrix form:

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1} & q_{n-1} \\ p_{n-2} & q_{n-2} \end{pmatrix} = ... = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} p_1 & q_1 \\ p_0 & q_0 \end{pmatrix}$$

The determinant gives: $p_n q_{n-1} - q_n p_{n-1} = (-1)^n = (-1)^{n-1}(p_1 q_0 - p_0 q_1) = (-1)^n$. □

Remark Since the a_n's are by definition strictly positive, we can easily see that (p_n) and (q_n) are increasing sequences. It can be shown recursively that

$$\frac{q_{n-1}}{q_n} = \frac{q_{n-1}}{\underbrace{a_n q_{n-1} + q_{n-2}}_{<1}} = \frac{1}{a_n + \frac{q_{n-2}}{q_{n-1}}} = ... = [0; a_n, a_{n-1} ..., a_2]$$

We will use this equality later in the proof.

Remark Looking at $[a_1; a_2, a_3, ...]$, we see that for odd n, the larger a_n is, the larger is the whole fraction, and for even n, the larger a_n is, the smaller is the whole fraction. If you don't see why immediately, try to write down the fraction and think first about the case where the continued fraction is finite. This observation will be crucial in the proof.

Proof. i) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $[a_1; a_2, ...]$ be its continued fraction, and $\frac{p_n}{q_n} = [a_1, ..., a_n]$ as before. Write $\alpha = [a_1; a_2, ..., a_n, \alpha_n] := \frac{\tilde{p_n}}{\tilde{q_n}}$. We can apply the recursive formula to $\frac{\tilde{p_n}}{\tilde{q_n}}$ to get $\alpha = \frac{\alpha_n p_n + p_{n-1}}{\alpha_n q_n + q_{n-1}}$. Using this formula and theorem 9, we get

$$\alpha - \frac{p_n}{q_n} = \frac{\pm 1}{r_n 1_n^2},\ r_n := \alpha_n + \frac{q_{n-1}}{q_n} = r_n = [a_{n+1}; a_{n+2} ...] + [0; a_n, a_{n-1}, ..., a_2]$$

Hence to prove i), it suffices to show that $r_n \geq \sqrt{5}$ for infinitely many n's. There are several cases:

Case 1: $a_n \geq 3$ for infinitely many n. Since $r_n = [a_{n+1}; a_{n+2} ...] + [0; a_n, a_{n-1}, ..., a_2] > a_{n+1}$, $r_n > 3 > \sqrt{5}$ for infinitely many n's.

Case 2: There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \in \{1, 2\}$

2.a) 2 appears infinitely many times. Then there exists infinitely many n such that $a_{n+1} = 2$ and $r_n = [2; a_{n+2}...] + [0; a_n, a_{n-1}, ..., a_2]$. $r'_n := [2; 2, 1, 2, 1, 2, ...] + [0; 2, 1, 2, 1, ..., a_2]$ is the minimal of all such numbers since for odd n, the smaller a_n the smaller the whole sequence, and the opposite holds for even n.

$$x := [0; 2, 1, 2, 1, ...] = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + ...}}} \text{ is the solution to } x = \cfrac{1}{2 + \cfrac{1}{1+x}}$$

Solving this equation, we get $x = -1 + \sqrt{3}$. Thus $r_n > r'_n > 2 + x = 1 + \sqrt{3} > \sqrt{5}$, and this holds for infinitely many n's.

2.b) 2 appears finitely many times, i.e. there exists $N \in \mathbb{N}$ such that $a_n = 1$ for all $n \geq N$. So for $n \geq N$,

$$r_n = [1; 1, 1, ...] + [0; a_n = 1, 1, ..., a_3, a_2]$$

Let $x = [1; 1, 1, ...], y_n := [0; a_n = 1, 1, ..., a_2]$.

$$x = 1 + \frac{1}{x} \implies x = \frac{1 + \sqrt{5}}{2}$$

$$y := \lim_{n \to \infty} y_n = [0; 1, 1, ...] = x - 1 = \frac{\sqrt{5} - 1}{2}$$

. $\implies \lim_{n\to\infty} r_n = x + y = \sqrt{5}$. Note that y_n alternate about y, i.e. if $y_n > y$ then $y_{n+1} < y$ (same reason as before for odd and even n's), so there are infinitely many n's for which $r_n > \sqrt{5}$. To prove ii), we will state a useful theorem without proof. ☐

Theorem 10 (Lagrange)**.** *Let α be an irrational number and $\frac{p_n}{q_n}$ one of its convergents. If $\theta < \frac{q_n}{q_n + q_{n-1}}$, then any $\frac{p}{q} \in \mathbb{Q}$ satisfying $\left|\alpha - \frac{p}{q}\right| < \frac{\theta}{q^2}$ is a convergent in the development of α. In particular, since q_n is increasing, this is always the case for $\theta < \frac{1}{2}$.*

Proof. ii) Let $\lambda > \sqrt{5}$. Then $\frac{1}{\lambda} < \frac{1}{2}$. By Langrange's theorem, every rational $\frac{p}{q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\lambda q^2}$ is a convergent of the form $\frac{p_n}{q_n}$. Then any irrational number whose continued fraction ends in $1, 1, ...$ is a counterexample. In fact, as shown in 2.b) of the previous proof, since $\lim_{n\to\infty} r_n = \sqrt{5}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $r_n < \lambda$, and so $\left|\alpha - \frac{p}{q}\right| = \frac{1}{r_n q_n^2} > \frac{1}{\lambda q_n^2}$. Hence, the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\lambda q_n^2}$ is only satisfied by a finite number of fractions. ☐

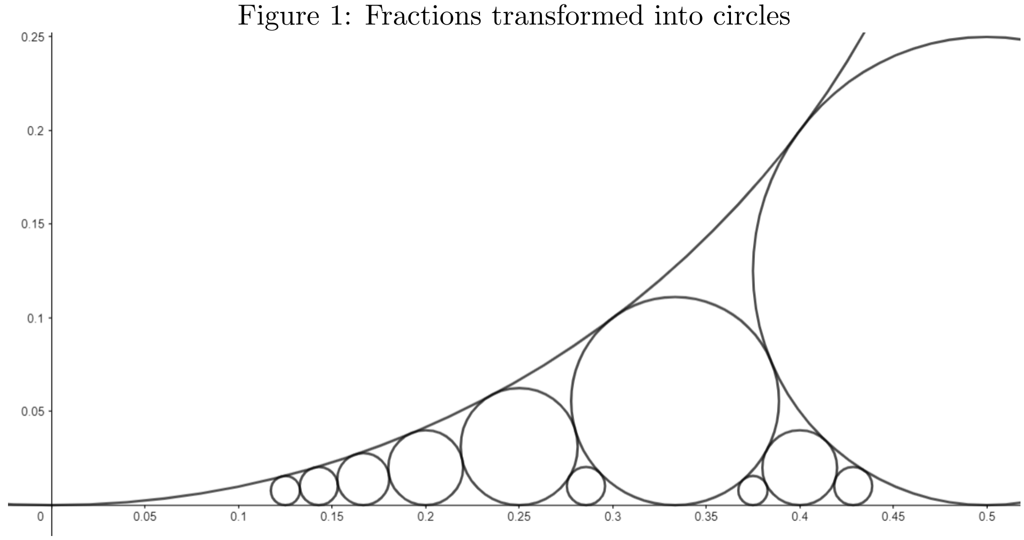
2 Geometric Proof

Now we will see another proof of Hurwitz's theorem using a completely different approach [For25]. The idea is to represent rational numbers as circles whose radii depend on the denominator of the fractions. This transforms the number theoretic problem into a geometric one. Now we will discuss the exact details of the construction, and it will quickly become evident why it is going to help us prove the theorem.

2.1 Construction of the circles

Through each rational point $x = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and the fraction is in its lowest terms ($(p, q) = 1$), we construct a circle of radius $\frac{1}{2q^2}$ tangent to the x-axis and lying in the upper half-plane. It is this circle, which will be the geometric picture of the fraction. The integers are represented by circles of radius $\frac{1}{2}$; the fractions $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}$, etc., by circles of radius $\frac{1}{18}$; and so on. Every small interval of the x-axis contains points of tangency of infinitely many of these circles.

Figure 1: Fractions transformed into circles



2.2 First findings

Lemma 2.1. *The representative circles of two distinct fractions are either tangent or external to one another.*

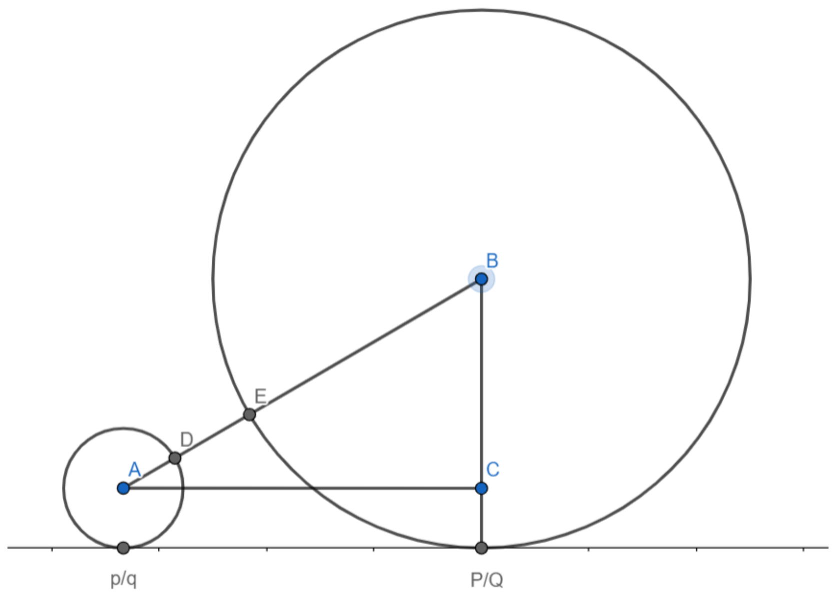
Proof. Let $\frac{p}{q} \neq \frac{P}{Q}$ and $(p, q) = 1 = (P, Q)$. Consider the distance between the centers of their representative circles. In the figure the horizontal distance AC is $\left|\frac{P}{Q} - \frac{p}{q}\right|$ and the vertical distance CB is $\left|\frac{1}{2Q^2} - \frac{1}{2q^2}\right|$. We have

$$\begin{aligned}
AB^2 &= \left(\frac{P}{Q} - \frac{p}{q}\right)^2 + \left(\frac{1}{2Q^2} - \frac{1}{2q^2}\right)^2 \\
&= \frac{(Pq - Qp)^2}{(Qq)^2} + (\frac{1}{2Q^2})^2 - \frac{1}{2q^2Q^2} + (\frac{1}{2q^2})^2 \\
&= \frac{(Pq - Qp)^2 - 1}{(Qq)^2} + (\frac{1}{2Q^2})^2 + \frac{1}{2q^2Q^2} + (\frac{1}{2q^2})^2 \\
&= \frac{(Pq - Qp)^2 - 1}{(Qq)^2} + \left(\frac{1}{2Q^2} + \frac{1}{2q^2}\right)^2 \\
&= \frac{(Pq - Qp)^2 - 1}{(Qq)^2} + (AD + EB)^2
\end{aligned}$$

We have to consider three cases:

- $|Pq - pQ| > 1 \Rightarrow AB > AD + EB$, and the two circles are external to one another.
- $|Pq - pQ| = 1 \Rightarrow AB > AD + EB$, and the two circles are tangent.
- $|Pq - pQ| < 1 \Rightarrow Pq - pQ = 0$ because $P, Q, p, q \in \mathbb{Z}$. Then $Pq = pQ$, which is a contradiction.

□



2.3 Adjacent fractions

Definition 2.1 (Adjacent fractions) Two fractions $\frac{p}{q}$ and $\frac{P}{Q}$ are adjacent if their representative circles are tangent. The condition for that is $|Pq - pQ| = 1$.

Lemma 2.2. *Each fraction $\frac{p}{q}$ possesses and adjacent fraction.*

Proof. For $\frac{p}{q}$ we have to find $\frac{P}{Q}$, such that $|Pq - pQ| = 1$

- Case 1 holds for $|q| = 1$, since $\frac{p}{1}$ has the adjacent fraction $\frac{p+1}{1}$
- Induction claim. All fractions whose denominators are less in absolute value than $|q|$ possess adjacent fractions.
- Induction step. Take any $\frac{p}{q}$. Pick $n \in \mathbb{Z}$ nearest to $\frac{p}{q}$. We can write

$$\frac{p}{q} = n + \frac{m}{q} = \frac{nq + m}{q}, 0 < |m| < |q|$$

$|m| < |q| \Rightarrow \frac{q}{m}$ has adjacent fraction $\frac{r}{s}$ (because of induction claim) $\Rightarrow |sq - rm| = 1$. Then the fraction

$$\frac{P}{Q} = n + \frac{s}{r} = \frac{nr + s}{r}, 0 < |m| < |q|$$

is adjacent to $\frac{p}{q}$ because

$$|Pq - pQ| = |(nr + s)q - (nq + m)r| = |sq - rm| = 1$$

□

Lemma 2.3. *If $\frac{P}{Q}$ is adjacent to $\frac{p}{q}$ then all fractions adjacent to $\frac{p}{q}$ are*

$$\frac{P_n}{Q_n} = \frac{P + np}{Q + nq}, n \in \mathbb{Z}$$

Corollary. *$\frac{P_n}{Q_n}$ and $\frac{P_{n+1}}{Q_{n+1}}$ are adjacent to each other.*

Proof of Corollary.

$$|(P + np)(Q + (n+1)q) - (P + (n+1)p)(Q + nq)| = |Pq - pQ| = 1. \quad \square$$

Proof. We find quite easily that the fractions given are adjacent to $\frac{p}{q}$ because

$$|(P + np)q - p(Q + nq)| = |Pq - pQ| = 1.$$

The circles corresponding to these fractions form a ring around the circle of $\frac{p}{q}$, all tangent to the circle of $\frac{p}{q}$ and each tangent to the circles which precede and follow it in the sequence. This ring completely surrounds the circle of $\frac{p}{q}$ because

$$\frac{P_n}{Q_n} = \frac{p}{q} + \frac{Pq - pQ}{q(Q + nq)} = \frac{p}{q} \pm \frac{1}{q^2(\frac{Q}{q} + n)}$$

When $n \to +\infty$, $\frac{P_n}{Q_n}$ approaches $\frac{p}{q}$ from one side; When $n \to -\infty$, $\frac{P_n}{Q_n}$ approaches $\frac{p}{q}$ from the other side.

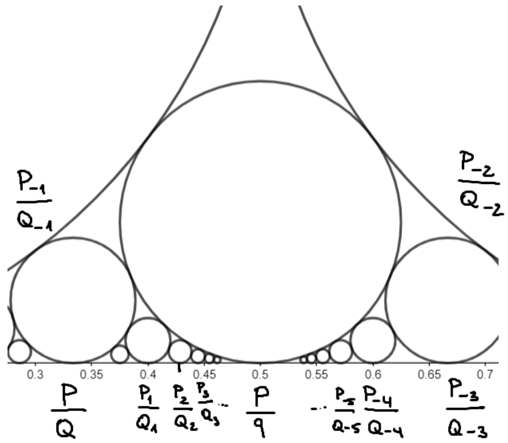
It is obvious from the geometric picture that it is not possible to draw another circle lying in the upper half-plane, touching the x-axis, and tangent to the circle of $\frac{p}{q}$, but not intersecting the circles of the ring surrounding the circle of $\frac{p}{q}$. It follows that there are no further fractions adjacent to $\frac{p}{q}$. □

Lemma 2.4. *Of the fractions adjacent to $\frac{p}{q}$ ($|q| > 1$) exactly two have denominators numerically smaller than q.*

Proof. It can be easily seen geometrically that for any circle there exist two bigger tangent circles.

We also can try to find values of n, such that $|Q + nq| < |q|$, equivalently $\left|n + \frac{Q}{q}\right| < 1$. Since $\frac{Q}{q} \notin \mathbb{Z} \Rightarrow \exists n_0, n_1$, such that $\left|n_0 + \frac{Q}{q}\right| < 1$ and $\left|n_1 + \frac{Q}{q}\right| < 1$ and $n_0 = n_1 + 1$.

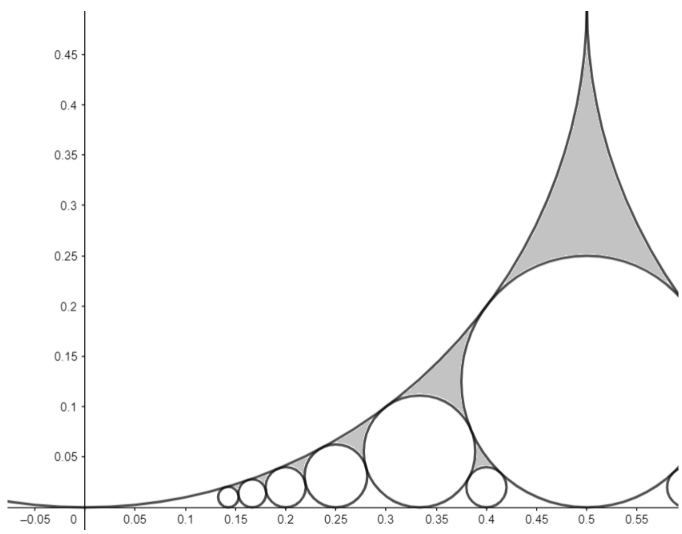
So there are exactly two distinct n values, such that $|Q + nq| < |q| \Rightarrow \frac{P+np}{Q+nq}$ is adjacent to $\frac{p}{q}$ and have smaller denominators. □



2.4 Mesh triangles and curves

Definition 2.2 (Mesh triangle) A mesh triangle is a part of the upper half plane exterior to all the circles of the system, which look like circular arc triangles

Remark Any two sides of a mesh triangle lie on circles belonging to adjacent fractions.



Definition 2.3 (Ford's Curve) A continuous curve starting with a point A_0 of the upper half-plane which remains in the upper half plane except that its terminal point, if any, may possibly lie on the x-axis.

Definition 2.4 (Ford's sequence) The fractions whose circles are passed through in succession by a Ford's curve.

Remark A fraction is not counted twice if the Ford's curve goes into a mesh triangle and returns to the same circle.

We also make the convention that if a Ford's curve touches two circles at their point of tangency without entering either then one or the other shall be considered as crossed by the Ford's curve (for example, the larger circle).

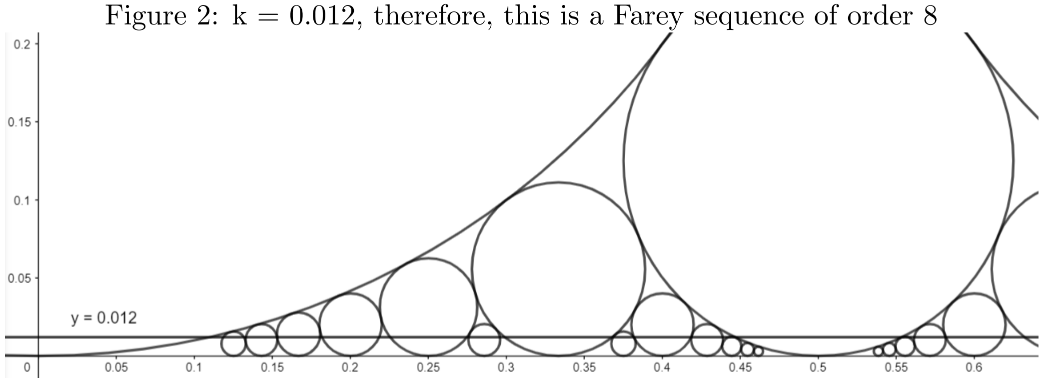
Corollary. *If two fractions are in succession in the Ford's sequence (a Ford's curve goes through them in succession) then the corresponding two fractions are adjacent.*

2.5 Farey sequence

Let L be a Ford's curve, which is a line $y = k$, parallel to the x-axis. Starting at $x = 0$ and stopping at $x = 1$. The points of tangency with the x-axis of the circles through which L passes are arranged in order from left to right; that is, the corresponding fractions are arranged in numerical order.

If $\frac{1}{(n+1)^2} < k < \frac{1}{n^2}$, L instersects the circles of all fractions in the interval $0 \leq x \leq 1$ whose denominators do not exceed n and the circles of no other fractions. These fractions arranged in numerical order constitute what is known as a Farey sequence of order n for the interval.

Figure 2: k = 0.012, therefore, this is a Farey sequence of order 8



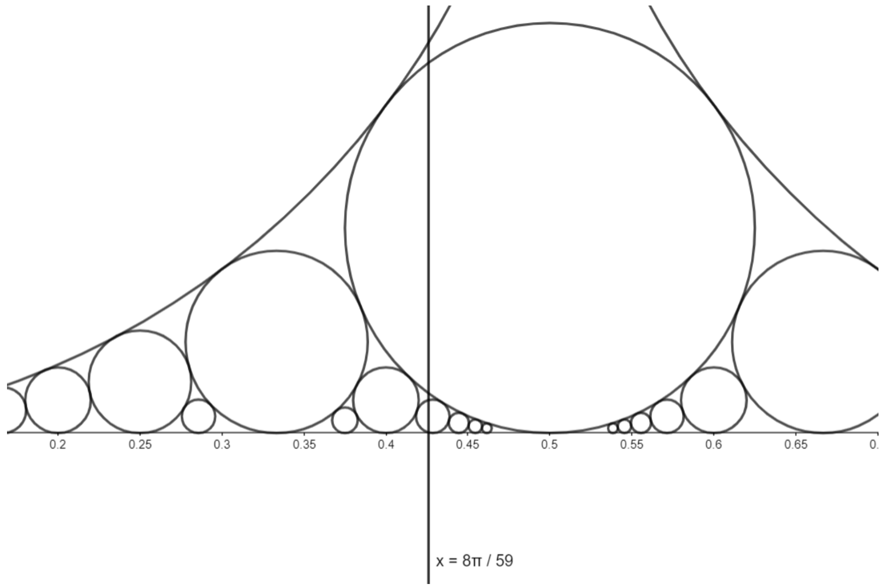
3 Geometric proof

3.1 Proof of k = 0.5

Proof. Let the Ford's curve L be a vertical line which terminates at the irrational α, so we have L is $x = \alpha$. Then L intersects infinitely many circles of the system we are considering and is tangent to none. If L cuts the circle of $\frac{p}{q}$ then the distance from α to $\frac{p}{q}$ is less than the radius:

$$\left| \frac{p}{q} - \alpha \right| < \frac{1}{2q^2}$$

The curve L thus provides us with an infinite suite of all thos fractions satisfying the equation when $k = \frac{1}{2}$ □



3.2 Proof of Hurwitz's theorem

The setup is identical to the previous one: we have a Ford's curve L, which is a vertical line terminating at the irrational α. It is quite evident that the line L cuts across infinitely many mesh triangles.

This is true because the only vertical lines which cross a finite amount of circles are the ones located at the rational points. The irrational ones by definition have to cross an infinite amount of circles.

Theorem 11. *Of the three fractions whose circles form the boundary of a mesh triangle which L crosses, at least one satisfies (1).*

Let $\frac{p_0}{q_0}$, $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$ be the fractions whose circles bound the mesh triangle area, where

$$0 < q_0 \leq q_1 < q_2 = q_0 + q_1, p_2 = p_0 + p_1$$

Without loss of generality, the largest of the three circles is on the right; that is $\frac{p_0}{q_0} > \frac{p_1}{q_1}$ and $q_1 \geq q_0$.

Let us define the point of contact between $\frac{p_0}{q_0}$ and $\frac{p_1}{q_1}$ as A. It divides the line of centers of these circles in the ratio of their radii - $\frac{1}{2q_1^2} : \frac{1}{2q_0^2}$ or $q_0^2 : q_1^2$. The abscissa of this point is

$$a = \frac{q_1^2 \frac{p_1}{q_1} + q_0^2 \frac{p_0}{q_0}}{q_1^2 + q_0^2} = \frac{p_1 q_1 + p_0 q_0}{q_1^2 + q_0^2}$$

The abscissas of B and C (the other vertices) may be written down by an interchange of letters,

$$b = \frac{p_1 q_1 + p_2 q_2}{q_1^2 + q_2^2}, \ c = \frac{p_0 q_0 + p_2 q_2}{q_0^2 + q_2^2}$$

In order that L cross the mesh triangle under consideration it is necessary and sufficient that α lie in the interval whose right end is c and whose left end is min(a,b). We find

$$a - \frac{p_1}{q_1} = \frac{q_0}{q_1(q_1^2 + q_0^2)}, \quad b - \frac{p_1}{q_1} = \frac{q_2}{q_1(q_1^2 + q_2^2)}$$

Now we subtract one from the other and we get

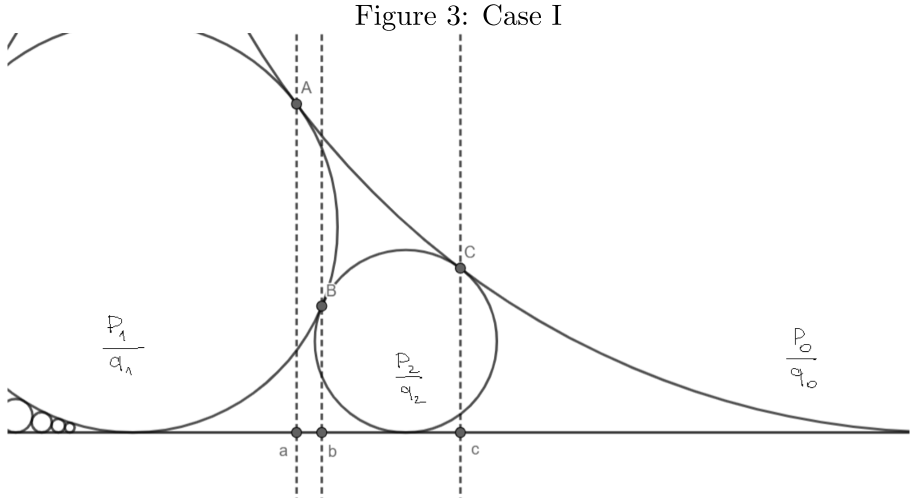
$$\begin{aligned} b - a &= \frac{q_2}{q_1(q_1^2 + q_2^2)} - \frac{q_0}{q_1(q_1^2 + q_0^2)} \\ &= \frac{q_2(q_1^2 + q_0^2) - q_0(q_1^2 + q_2^2)}{q_1(q_1^2 + q_2^2)(q_1^2 + q_0^2)} \\ &= \frac{(q_0 + q_1)(q_1^2 + q_0^2) - q_0(q_1^2 + (q_0 + q_1)^2)}{q_1(q_1^2 + q_2^2)(q_1^2 + q_0^2)} \\ &= \frac{q_1^2 - q_1 q_0 - q_0^2}{(q_1^2 + q_2^2)(q_1^2 + q_0^2)} \end{aligned}$$

Now we can define $s = \frac{q_1}{q_0} \geq 1$. To figure out whether a or b is smaller we have to find out the sign of $q_1^2 - q_1 q_0 - q_0^2$. This is equivalent to finding out the sign of $\frac{q_1^2}{q_0^2} - \frac{q_1}{q_0} - 1 = s^2 - s - 1$ (we divided it by q_0^2). Now, by factoring we get

$$s^2 - s - 1 = \left(s + \frac{\sqrt{5} - 1}{2}\right)\left(s - \frac{\sqrt{5} + 1}{2}\right)$$

The first factor is obviously positive, so the sign is determined by the second factor. We have two cases to consider.

Figure 3: Case I



Case I. $a < b$ and $s > \frac{\sqrt{5}+1}{2}$. We will show that $\frac{p_0}{q_0}$ satisfies the inequality with $k = \frac{1}{\sqrt{5}}$. We have

$$\left|\alpha - \frac{p_0}{q_0}\right| < \frac{p_0}{q_0} - a = \frac{p_0}{q_0} - \frac{p_1 q_1 + p_0 q_0}{q_1^2 + q_0^2} = \frac{q_1}{q_0(q_1^2 + q_0^2)} = \frac{s}{s^2 + 1} \frac{1}{q_0^2}$$

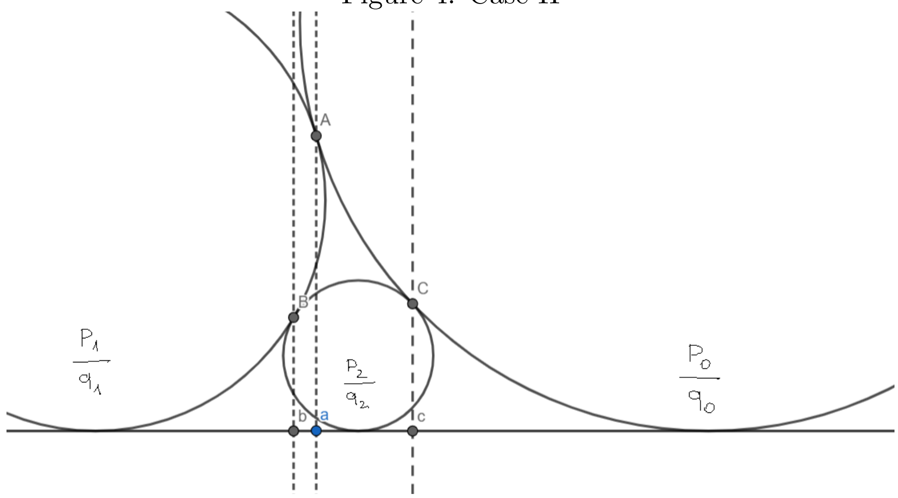
If $\frac{s}{s^2+1} \leq \frac{1}{\sqrt{5}}$, then we have what we want. Therefore, we have to prove that $\frac{s}{s^2+1} > \frac{1}{\sqrt{5}}$ is impossible.

Let

$$\frac{s}{s^2+1} > \frac{1}{\sqrt{5}} \Rightarrow s^2 - \sqrt{5}s + 1 < 0 \Rightarrow \left(s - \frac{\sqrt{5}+1}{2}\right)\left(s - \frac{\sqrt{5}-1}{2}\right) < 0$$

However, that is impossible because $s > \frac{\sqrt{5}+1}{2}$.

Figure 4: Case II



Case II. $a > b$ and $s < \frac{\sqrt{5}+1}{2}$. We will show that $\frac{p_2}{q_2}$ satisfies the inequality with $k = \frac{1}{\sqrt{5}}$. It is clear that c is nearer $\frac{p_2}{q_2}$ than b is, since C is higher on the circle than B.

$$\left|c - \frac{p_2}{q_2}\right| < \left|b - \frac{p_2}{q_2}\right|$$

Then

$$\left|\alpha - \frac{p_2}{q_2}\right| < \frac{p_2}{q_2} - b = \frac{p_2}{q_2} - \frac{p_1q_1 + p_2q_2}{q_1^2 + q_2^2} = \frac{q_1}{q_2(q_1^2 + q_2^2)} = \frac{s(s+1)}{s^2 + (s+1)^2}\frac{1}{q_2^2}$$

If $\frac{s}{s^2+1} \leq \frac{1}{\sqrt{5}}$, then we have what we want. Therefore, we have to prove that $\frac{s}{s^2+1} > \frac{1}{\sqrt{5}}$ is impossible.

Let

$$\frac{s(s+1)}{s^2+(s+1)^2} > \frac{1}{\sqrt{5}} \Rightarrow$$
$$(\sqrt{5}-2)s^2 + (\sqrt{5}-2)s - 1 > 0 \Rightarrow$$
$$s^2 + s - (\sqrt{5}+2) > 0 \Rightarrow$$
$$\left(s - \frac{\sqrt{5}+1}{2}\right)\left(s + \frac{\sqrt{5}+1}{2}\right) < 0$$

However, that is impossible because $s < \frac{\sqrt{5}+1}{2}$, therefore, the first fraction is negative and the second is positive.

Lemma 3.1. *For* $\alpha = \frac{1+\sqrt{5}}{2}$ *there exist only a finite number of* $\frac{p}{q} \in \mathbb{Q}$ *such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

Proof. Let $0 < h < 1 h \in \mathbb{R}$ and $|\theta| < h < 1$

$$\left|\frac{p}{q} - \frac{\sqrt{5}+1}{2}\right| < \frac{h}{\sqrt{5}q^2}$$
$$\frac{p}{q} - \frac{\sqrt{5}+1}{2} = \frac{\theta}{\sqrt{5}q^2}$$
$$\frac{p}{q} - \frac{1}{2} = \frac{\sqrt{5}}{2} + \frac{\theta}{\sqrt{5}q^2}$$
$$5q^2((p^2 - pq - q^2) - \theta) = \theta^2$$

$p^2 - pq - q^2 \in \mathbb{Z}$ and $p^2 - pq - q^2 \neq 0$ because otherwise $\frac{p}{q}^2 - \frac{p}{q} - 1 = 0 \Rightarrow \frac{p}{q} \in \mathbb{R} \setminus \mathbb{Q}$ and it also must be positive.

$$p^2 - pq - q^2 \geq 1$$
$$q^2 = \frac{\theta^2}{5((p^2 - pq - q^2) - \theta)} < \frac{h^2}{5(1-h)}$$

□

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