Commutative Algebra

HS 2021

(E. Kowalski)

Organization:

(1) Home page

https://metaphor.ethz.ch/x/2021/hs/401-3132-00L/

(2) Forum

https://forum.math.ethz.ch/c/autumn-2021/commutative-algebra/89

(3) Book

"(Mostly) commutative algebra", A. Chambert Loir

https://webusers.imj-prg.fr/~antoine.chambert-loir/publications/teach/commalg.pdf

(4) The lecture happens at the blackboard.
The sound will be recorded / streamed.
The notes and the reference book
are available to complete the recording.

(5) Information on the exercises on the
home page.
Due to the large number of students, there
might be only one new exercise sheet
every two weeks.

Outline

(1) Introduction / Motivation / Reminders

(2) The language of categories and functors

(3) Constructions of rings (quotients, polynomial rings, localization)

(4) Noetherian rings and modules

(5) Tensor product, multilinear algebra

(6) Krull dimension

(7) Integral extensions

(8) Finitely generated algebras over fields

(9) Artinian rings and modules

(10) Primary decomposition

(11) Discrete valuation rings

(Some adjustments likely depending on time available)

Chapter I

Introduction

1. What is commutative algebra?

It is "the study of commutative rings (with unit)", which means also the study of their ideals and of modules over such rings.

Convention: unless otherwise specified, all rings R in this lecture are commutative and have a unit 1_R s.t.

$$\forall a \in R, \qquad a \cdot 1_R = 1_R \cdot a = a$$

(e.g. $2\mathbb{Z} \subset \mathbb{Z}$ is not a ring)

and all ring morphisms

$$f : R \longrightarrow S$$

preserve the unit : $f(1_R) = 1_S$

(e.g. $\{0\} \subset \mathbb{Z}$ is not a morphism).

2. Why study commutative algebra?

For many, this is because it is the foundations for modern algebraic geometry, which is the study of solution sets of systems of polynomial equations, hence a natural (unavoidable) generalization of linear algebra.

[e.g. $2x + 3y - z = 1$ defines a "linear" object

"algebraic set" $2x^3 + 3y^2 - xyz = 1$ defines a non-linear one]

There are however many other reasons to study commutative algebra, and in particular ideas like the tensor product of modules over rings are universally useful.

3. What are sample theorems of commutative algebra?

The following results do not explicitly mention any specific objects of commutative algebra, but show what it can do.

Th. 1. (Hilbert / Noether)

Let $n \geq 1$ be an integer.

Let $(P_i)_{i \in I}$ be any collection of polynomials in $\mathbb{C}[X_1, \dots, X_n]$. There exists a finite set B of polynomials such that

$$\{x = (x_1, \dots, x_n) \in \mathbb{C}^n \,/\, P_i(x) = 0 \text{ for all } i\}$$
$$\|$$
$$\{x = (x_1, \dots, x_n) \in \mathbb{C}^n \mid P(x) = 0 \text{ for all } P \in B\}$$

Ex. Think / prove the case where all P_i have degree ≤ 1.

Th. 2. (Hilbert's Nullstellensatz)

Let $n \geq 1$, $m \geq 0$ be integers.

Let $P_1, \dots, P_m$ be elements of $\mathbb{C}[X_1, \dots, X_n]$.

There exists a common zero $x \in \mathbb{C}^n$ of $P_1, \dots, P_m$ [i.e. $P_i(x) = 0$, $1 \leq i \leq m$] <u>unless</u> there exist polynomials Q_i s.t.

$$P_1 Q_1 + \dots + P_n Q_n = 1.$$

<u>Remark.</u> (1) Note that if such a relation exists, then conversely there cannot be a common zero of these polynomials, since we would get the contradiction $0 = 1$.

(2) This statement does not extend to $\mathbb{R}$ instead of $\mathbb{C}$:

$$X^2 + Y^2 + 1 = 0$$

has no solution $(x, y) \in \mathbb{R}^2$, but there is no Q s.t. $(X^2 + Y^2 + 1) Q = 1$.

Ex. again check the case when all P_i have degree ≤ 1.

Th. 3 - (Skolem - Mahler - Lech)

Consider a sequence $(u_n)_{n\geq 0}$ of complex numbers satisfying a linear recurrence relation:

$$u_{n+k} + a_1 u_{n+k-1} + \dots + a_k u_n = 0$$

$(a_i \in \mathbb{C})$ for some $k \geq 1$.

The set $N = \{ n \geq 0 \mid u_n = 0 \}$ is the union of finitely many sets of the form $A_{ab} = \{ ak + b \mid k \geq 0 \}$ (where $a = 0$ is allowed).

Ex. look at the cases

$$u_{n+2} - u_{n+1} - u_n = 0$$

$$u_{n+4} - u_{n+2} - u_n = 0.$$

4 – Some "guidelines"

Commutative algebra is necessarily an abstract subject. There are many concepts / definitions and these (and the proofs) are often far from transparent at first sight, and may seem to have very little motivation.

The following can help to keep a motivating picture. Quite a few times, the goal is to obtain some <u>analogue</u> of known properties of very well understood / well-behaved objects. We will see many cases of definitions of

(1) properties of modules over a ring R that approach those of <u>vector spaces over a field</u>.

(2) properties of rings that approach those of the simplest rings: fields, principal ideal domains.

5. Some reminders / definitions

Definition (algebra over a ring).

Let R be a ring [commutative with unit]

An **algebra over R** (or R-algebra) is a ring S **with** a ring morphism

$$\varphi : R \longrightarrow S$$

(called the "structure morphism").

If $\psi : R \longrightarrow T$ is another R-algebra, then an **R-morphism** $f : S \longrightarrow T$ is a ring morphism s.t. $f(\varphi(r)s) = \psi(r)f(s)$ for all $r \in R$, $s \in S$. (In other words, the diagram

$$\begin{tikzcd} & S \arrow[d, "f"] \\ R \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & T \end{tikzcd}$$

commutes.)

Remarks: φ, ψ are often left unspecified and one writes $r \cdot s = \varphi(r)s$ for $r \in R$ and $s \in S$; then $f(r \cdot s) = r \cdot f(s)$ ...

Examples :

(1) $\mathbb{Z}/_{n}\mathbb{Z}$ is a $\mathbb{Z}$-algebra with structure morphism $\mathbb{Z} \longrightarrow \mathbb{Z}/_{n}\mathbb{Z}$ being reduction modulo n.

(2) $\mathbb{C}$ is an $\mathbb{R}$-algebra with the inclusion morphism $\mathbb{R} \hookrightarrow \mathbb{C}$.

(3) For a ring R, $R[X]$ is an R-algebra with the morphism $R \hookrightarrow R[X]$ (constant polynomials)

(4) = (1 bis) Any ring R is a $\mathbb{Z}$-algebra via the (unique) morphism

$$\underset{\mathbb{Z}}{\overset{\in}{1}} \longmapsto 1_R$$

(5) = (2 bis) For any field extension E/F, E is an F-algebra.

(6) Given R and S, there may be more than one structure of R-algebra on S.

For instance: $\mathbb{C}$ is a $\mathbb{C}$-algebra either

with $z \cdot w = zw$

or

$z \cdot w = \bar{z} \cdot w$.

Reminders:

(1) $\mathbb{N} = \{0, 1, \dots\}$

(2) A module M over a ring R is an abelian group with a multiplication

$$R \times M \longrightarrow M$$

s.t.
$$\begin{cases} (r_1 + r_2) m = r_1 m + r_2 m, \\ 0_R \cdot m = 0 , \quad 1 \cdot 0_M = 0_M \\ 1_R \cdot m = m \\ r(m_1 + m_2) = r m_1 + r m_2 \\ (r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m). \end{cases}$$

(3) An ideal $I \subset R$ is a submodule of R.

[Ex. if R field then R-module $= R$-vector space]

(4) An ideal $I \subset R$ is :

<u>prime</u> $\iff$ R/I is integral domain

$\iff$ $I \neq R$ and ($ab \in I \iff a \in I$ or $b \in I$)

<u>maximal</u> $\iff$ R/I is a field

$\iff$ $I \neq R$ and there is no ideal $J \subset R$

s.t. $I \subsetneq J \subsetneq R$

Two fundamental properties:

(i) Given $f: R \to S$, and $P \subset S$ prime, the ideal $f^{-1}(P) \subset R$ is prime

(ii) [Krull] Let $I \neq R$ be an ideal; there exists a maximal ideal M s.t.

$$I \subset M \; (\subsetneq R)$$

Ex. If $f: R \to S$ and $M \subset S$ is maximal, then $f^{-1}(M) \subset R$ is prime but not always maximal. (E.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$)

(5) $R^\times$ = group of units of R

(6) $x \in R$ is called <u>nilpotent</u> if there exists $n \geq 1$ s.t. $x^n = 0$.

(7) A ring R is <u>reduced</u> if it contains no non-zero nilpotent elements.