Chapter II

The language of categories and functors

[Ref. ACL
Appendix A.3]

1. Categories and functors

In mathematics, we often define new kinds of objects [e.g. groups, vector spaces] and then special maps between them to discuss their relations.

It becomes useful to have a common language to discuss general properties of any such construction; these are categories. And since categories also "interact", we have "functors" between them.

Definition (Category)

A category $\mathcal{C}$ is the data of

(i) a "collection" of "objects" of $\mathcal{C}$

(ii) for any objects X and Y of $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms [in $\mathcal{C}$] from X to Y; an element $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is often denoted

$$X \xrightarrow{f} Y$$

(iii) for any objects X, Y, Z of $\mathcal{C}$, a map of sets

$$\begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z) \\ (g, f) \longmapsto f \circ g \end{cases}$$

[i.e.
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & Y \arrow[r, "f"] & Z \end{tikzcd}$$
]

and for any object X of $\mathcal{C}$, an element $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$

such that :

$$Id_Y \circ f = f \qquad X \xrightarrow{f} Y \xrightarrow{Id_X} Y$$

$$g \circ Id_X = g \qquad X \xrightarrow{Id_X} X \xrightarrow{g} Y$$

$$f \circ (g \circ h) = (f \circ g) \circ h$$

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[rr, bend right, "g \circ h"'] & X \arrow[r, "g"] \arrow[rr, bend left, "f \circ g"] & Y \arrow[r, "f"] & Z
\end{tikzcd}$$

__Remark.__ What do we mean by "collection"?

In the simplest case, just a set.

But we also want to speak of "the category of sets", with __all__ sets as objects. Since there is no "set of all sets", this creates a difficulty.

For us, this "collection" simply means that there is some logical formula $\Phi_{\mathcal{C}}(X)$ with one (set as) variable such that an object of $\mathcal{C}$ is a set such that $\Phi_{\mathcal{C}}(X)$

is true.

Examples :

(1) $\mathscr{C} = (\mathrm{Set})$, the category of sets, with morphisms the usual maps of sets.

(2) $\mathscr{C} = (\mathrm{Grp})$, the category of groups, with morphisms the usual morphisms ~~of groups~~.

[Here $\Phi_{\mathscr{C}}(\{G, e, m, i\}) =$

" $e \in G$ and $m : G \times G \to G$ and $i : G \to G$ and $(\forall g \in G,\ m(g,e) = m(e,g) = g)$ and

$(\forall g \in G,\ m(g, i(g)) = m(i(g), g) = e)$ and

$(\forall g \forall h \forall k,\ m(g, m(h,k)) = m(m(g,h),k))$ "]

(3) Given a ring R, $\mathscr{C} = (\mathrm{Mod})_R$, the category of R-modules

(4) The category (Rings) of rings; the category $(\mathrm{Alg})_R$ of R-algebras.

(5) Less obvious: given a ring R, the category $\mathscr{C} = (\mathrm{Maps})_R$ of R-linear

maps :

- objects : $\left(M \xrightarrow{f} N \right)$

R - linear

R - modules

- morphisms :

$$\mathrm{Hom}_{\mathcal{C}}\left(M \xrightarrow{f} N ,\ M' \xrightarrow{g} N' \right)$$

$$= \left\{ (\varphi, \psi) \ \middle/ \ \begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi"'] & N \arrow[d, "\psi"] \\ M' \arrow[r, "g"] & N' \end{tikzcd} \ \text{commutes, i.e. } \psi \circ f = g \circ \varphi \right\}$$

with

$$\mathrm{Id}_f = \left(\mathrm{Id}_M , \mathrm{Id}_N \right) \qquad \begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & N \arrow[d, "\mathrm{Id}"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

and composition

$$(\varphi', \psi') \circ (\varphi, \psi) = (\varphi' \circ \varphi , \psi' \circ \psi) \qquad \begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi"'] & N \arrow[d, "\psi"] \\ M' \arrow[r] \arrow[d, "\varphi'"'] & N' \arrow[d, "\psi'"] \\ M'' \arrow[r] & N'' \end{tikzcd}$$

[Ex. check that this is a category]

Definition [Functor]

Let $\mathcal{C}$, $\mathcal{D}$ be categories. A **functor** F from $\mathcal{C}$ to $\mathcal{D}$, denoted

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

is a rule assigning

(i) an object $F(X)$ of $\mathcal{D}$ for any object X of $\mathcal{C}$

(ii) a morphism $F(f) : F(X) \longrightarrow F(Y)$

to any morphism $f : X \longrightarrow Y$ of $\mathcal{C}$

such that:

$$F(Id_X) = Id_{F(X)}$$

$$F(f \circ g) = F(f) \circ F(g)$$

for all $X \xrightarrow{g} Y \xrightarrow{f} Z$ in $\mathcal{C}$.

$$\left[F(X) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(Z) \right]$$

Examples - (1) There is a "forgetful" functor

$$\begin{cases} (Grp) \longrightarrow (Sets) \\ G \longmapsto G \\ f \longmapsto f \end{cases}$$

and similarly, e.g

$$\begin{cases} (\mathrm{Mod})_R \longrightarrow (\mathrm{Grp}) \\ M \longmapsto (M,+) \\ f \longmapsto f \end{cases}$$

or

$$\begin{cases} (\mathrm{Alg})_R \longrightarrow (\mathrm{Rings}) \\ (\varphi : R \to S) \longmapsto S \\ f \longmapsto f \end{cases}$$

(2) There is a functor "units"

$$u \begin{cases} (\mathrm{Rings}) \longrightarrow (\mathrm{Grps}) \\ R \longmapsto R^\times \\ f \longmapsto f|R^\times \end{cases}$$

To check that this is well-defined, we must check that if $r \in R$ is invertible then $f(r) \in S$ is in $S^\times$: but

$$xy = 1_R \Rightarrow f(x) f(y) = 1_S$$

$$\Rightarrow f(x) \in S^\times .$$

(3) Consider the category $(\mathrm{Top}\text{-}\mathrm{pt})$ of pairs (X, x_0) of a topological space X and a "base point" $x_0 \in X$,

with $f : (X, x_0) \longrightarrow (Y, y_0)$ being a continuous map $f: X \longrightarrow Y$ s.t. $f(x_0) = y_0$.

The <u>fundamental group</u> defines a functor

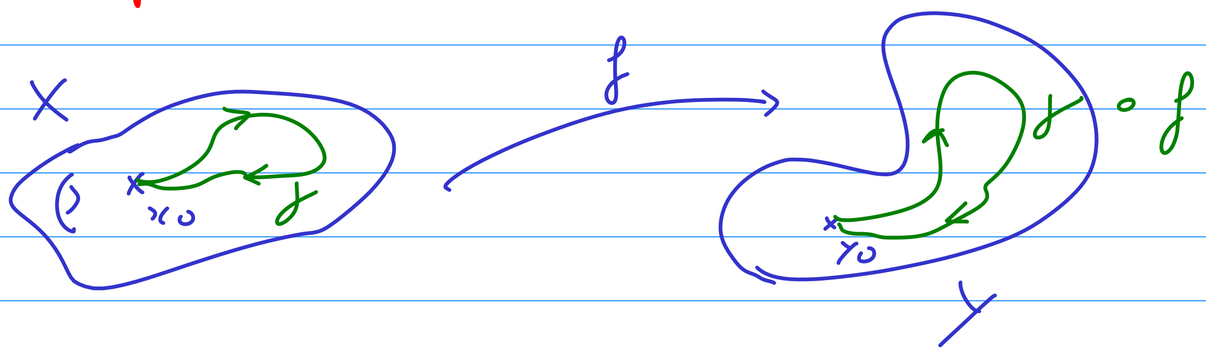
$$\pi_1 : \begin{cases} (\text{Top-pt}) \longrightarrow (\text{Grps}) \\ (X, x_0) \longmapsto \pi_1(X, x_0) \end{cases}$$

where the functor on morphisms is induced by

$$\gamma \longmapsto \gamma \circ f \;, \quad \text{where}$$

$$\gamma : [0,1] \xrightarrow{\gamma} X$$

is a loop based at x_0.



(4) Let K be a field.

Defined $(\text{Vect})_K = (\text{Mod})_K$ to be the category of K-vector spaces.

We want to define a functor F where

$F(V) = V' = \text{Hom}_K(V, K)$ is the <u>dual</u>

vector space. This is a natural construction but note that there is no corresponding way to send "naturally" a linear map $V \xrightarrow{f} W$ to a linear map $V' \longrightarrow W'$. Rather, the transpose gives a linear map $W' \xrightarrow{{}^t f} V'$, where the arrow was reversed and then we have

$$ {}^t(f \circ g) = {}^t g \circ {}^t f. $$

This means that F, with $F(f) = {}^t f$, is a <u>contravariant</u> functor.

[<u>Def.</u> [Opposite category, contravariant functor]

Let $\mathcal{C}$ be a category.

The <u>opposite category</u> $\mathcal{C}^{opp}$ is the category with the same objects as $\mathcal{C}$ and

$$ \mathrm{Hom}_{\mathcal{C}^{opp}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X) $$

and $f \circ_{\mathcal{C}^{opp}} g = g \circ f$

$\underline{\underline{\mathcal{C}^{opp}}}$ $\qquad\qquad\qquad$ $\underline{\underline{\mathcal{C}}}$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend right, "f \circ_{\mathcal{C}^{opp}} g"'] & Y \arrow[r, "f"] & Z \end{tikzcd} \qquad\qquad \begin{tikzcd} X & Y \arrow[l, "g"'] & Z \arrow[l, "f"'] \arrow[ll, bend left, "g \circ f"] \end{tikzcd}$$

Let $\mathcal{D}$ be a category. A <u>contravariant</u> functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}^{opp}$$

[so $F(X)$ object of $\mathcal{D}$ for all X and

$$F(X \xrightarrow{f} Y) = \left(F(X) \xleftarrow{F(f)} F(Y) \right) \quad]$$

<u>Examples</u> : let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$.

There are then two functors attached to X :

(1) $h^X : \mathcal{C} \longrightarrow (Sets)$ $\qquad\qquad X \xrightarrow{g} Y \xrightarrow{f} Z$

$$Y \longmapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

$$f \longmapsto \underbrace{(g \longmapsto f \circ g)}_{\mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(X,Z)}$$

(2) $h_X : \mathcal{C} \longrightarrow (Sets)^{opp}$ (contravariant)

$$Y \longmapsto Hom_{\mathcal{C}}(Y, X)$$

$$Z \xrightarrow{f} Y \longmapsto (g \longmapsto g \circ f)$$

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[rr, bend right, "h_X(f)(g)"'] & Y \arrow[r, "g"] & X \end{tikzcd}$$

Check that this is contravariant :

$$h_X(f_1 \circ f_2)(g) = g \circ (f_1 \circ f_2)$$

$$(h_X(f_2) \circ h_X(f_1))(g) = h_X(f_2)(g \circ f_1)$$

$$= g \circ f_1 \circ f_2$$

$$(W \xrightarrow{f_2} Z \xrightarrow{f_1} Y) \xrightarrow{g} X$$

2 – Isomorphisms

One advantage of categories is that it clarifies the notion of isomorphism for any mathematical structure.

Def. Let $\mathcal{C}$ be a category and

$$X \xrightarrow{f} Y$$

a morphism in $\mathcal{C}$. One says that f

is an <u>isomorphism</u> if there exists

$$Y \xrightarrow{g} X$$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

in $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that

$$f \circ g = \mathrm{Id}_Y, \quad g \circ f = \mathrm{Id}_X$$

<u>Lemma</u> - If $f : X \longrightarrow Y$ is an isomorphism, then the g above is unique.

<u>Proof</u>. Suppose $\begin{cases} f \circ h = \mathrm{Id}_Y \\ h \circ f = \mathrm{Id}_X . \end{cases}$

Then

$$\begin{aligned} g = g \circ \mathrm{Id}_Y &= g \circ (f \circ h) \\ &= (g \circ f) \circ h \\ &= \mathrm{Id}_X \circ h = h . \end{aligned}$$

$\square$

<u>Def</u>. g is called the <u>inverse</u> of f, and denoted f^{-1}.

The inverse $f^{-1} : Y \longrightarrow X$ is also an isomorphism and $(f^{-1})^{-1} = f$.

<u>Lemma</u>. If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor and $f : X \to Y$ is an isomorphism, then $F(f) : F(X) \longrightarrow F(Y)$ is one, and

$$F(f)^{-1} = F(f^{-1}).$$

<u>Proof</u>.

$$\begin{aligned} F(f) \circ F(f^{-1}) &= F(f \circ f^{-1}) \\ &= F(Id_Y) \\ &= Id_{F(Y)} \end{aligned}$$

and $F(f^{-1}) \circ F(f) = Id_{F(X)}$.

$\square$

<u>Ex.</u> If $R \xrightarrow{\sim} S$ is an isomorphism of rings, then $R^\times \xrightarrow{\sim} S^\times$.

<u>Terminology/notation</u>: $Isom_{\mathcal{C}}(X, Y) = \{\text{isom. } X \to Y\}$

$Aut_{\mathcal{C}}(X) = Isom_{\mathcal{C}}(X, X)$ ("automorphisms")

<u>Lemma</u>. $Aut_{\mathcal{C}}(X)$ is a group with composition.

3 - Yoneda's Lemma

Here is a first non-trivial result about categories: the functor h_X [s.t.

$$h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)\ ]$$

characterizes an object X up to a "canonical" isomorphism.

Def. F, G functors $\mathcal{C} \longrightarrow \mathcal{D}$

A natural transformation

$$\Phi : F \longrightarrow G$$

is a rule assigning to every object X a morphism $\Phi(X) : F(X) \longrightarrow G(X)$ s.t. for any $X \xrightarrow{f} Y$ the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\Phi(X)"'] & F(Y) \arrow[d, "\Phi(Y)"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

commutes. It is an isomorphism if

$\Phi(X)$ is an isomorphism for all X.

Example. Let $X \xrightarrow{f} Y$ be a morphism.
Then $\Phi_f(X) = (g \mapsto f \circ g)$
is a map

$$\begin{array}{ccc} \mathrm{Hom}(Z, X) & \longrightarrow & \mathrm{Hom}(Z, Y) \\ \| & & \\ h_X(Z) & \longrightarrow & h_Y(Z) \end{array}$$

and one checks that Φ_f is a natural transformation of functors $\mathcal{C} \longrightarrow (\mathcal{E}ns)^{opp}$.

Conversely, let $\Phi : h_X \longrightarrow h_Y$ be a natural transformation: for $W \xrightarrow{g} Z$, we get

$$\begin{tikzcd} \mathrm{Hom}(Z, X) \arrow[r, "h_X(g)"] \arrow[d, "\Phi(Z)"'] & \mathrm{Hom}(W, X) \arrow[d, "\Phi(W)"] \\ \mathrm{Hom}(Z, Y) \arrow[r, "h_Z(g)"] & \mathrm{Hom}(W, Y) \end{tikzcd}$$

Take $Z = X$, and $Id_X \in \mathrm{Hom}(Z, X)$;
let $f = \Phi(X)(Id_X) \in \mathrm{Hom}(X, Y)$:

$$X \xrightarrow{f} Y$$

We show that $\Phi = \Phi_f$.

Indeed, let Z be an object of $\mathcal{C}$. We want to compute the map $\Phi(Z) : \mathrm{Hom}(Z, X) \longrightarrow \mathrm{Hom}(Z, Y)$

Use the diagram with $g : Z \to X$:

$$\begin{array}{ccc}
\mathrm{Hom}(X, X) & \longrightarrow & \mathrm{Hom}(Z, X) \\
Id_X & \longmapsto & Id_X \circ g = g \\
\Phi(X) \downarrow \quad \downarrow & & \downarrow \quad \downarrow \Phi(Z) \\
f & & \Phi_Z(g) \\
\mathrm{Hom}(X, Y) & \longrightarrow & \mathrm{Hom}(Z, Y) \\
f & \longmapsto & f \circ g
\end{array}$$

shows that
$$\boxed{\Phi_Z(g) = f \circ g}$$

<u>Corollary</u> - If h_X is isomorphic to h_Y, then X is isomorphic to Y, and conversely.

4 - Final remarks

(1) Do not underestimate categories!

Ex. "tannakian categories" give ways to define groups in some contexts in which no other construction is known.

(2) Do not overestimate categories!

(They usually are combined with lots of other mathematical insights to give results in other fields)

(3) Be precise about which category you work in!

[A given object often belongs to more than one category and the morphisms are not the same!

eg $\mathbb{R}$ is
- a set
- a topological space
- an abelian group
- a topological group
- ...

]