Chapter IV [ACL 6.3]

Noetherian rings and modules

Goal: the noetherian property is a fundamental finiteness property in commutative algebra, which distinguishes rings / modules sharing some of the simplest properties of fields / vector spaces.

1 - Definition

If K is a field and V a finite-dim. vector space over K, then any subspace $W \subset V$ is also finite-dim., and has smaller (or equal) dimension. These fundamental facts fail in general, even for free R-modules of rank 1:

(1) R itself is free over R of rank $\underline{1}$

(2) but its submodules $I \subset R$, ideals of R, may not be finitely generated, or free, or

if they are finitely generated, may not be principal.

Ex. Let $R = \mathbb{C}[x_1, \_\_\_, x_n, \_\_\_]$ be a polynomial ring in infinitely many variables. Then the ideal

$$I = (x_1, \dots, x_n, \dots)$$

generated by the variables is not finitely generated.

Definition (Noetherian module / ring)

R ring, M an R-module.

The module M is noetherian if any sub-module $N \subset M$ is finitely-generated.

(In particular, $M \subset M$ is itself finitely-generated)

The ring R is noetherian if R is noetherian as an R-module, i.e. all ideals $I \subset R$ are finitely-generated.

Ex. Fields and P.I.Ds are noetherian.

Any finite-dim. vector space over a field, or finitely-generated module over a PID, is noetherian.

The ring $\mathbb{C}[(x_n)_{n\geq 1}]$ is not noetherian.

(ACL, 6.3.2)

Proposition. R ring, M R-module.

The following conditions are equivalent:

(i) M is noetherian

(ii) For any non-empty set $\mathcal{X}$ of R-submodules of M, there exists a maximal element $N \in \mathcal{X}$, i.e. a submodule $N \in \mathcal{X}$ such that no element N' of $\mathcal{X}$ satisfies $N \subset N'$ and $N \neq N'$.

(iii) If $N_0 \subset N_1 \subset \dots \subset N_n \subset \dots$ is a sequence of submodules of M, there there is some $n_0 \geq 0$ s.t. $N_n = N_{n_0}$ if $n \geq n_0$.

(One says that any increasing sequence of submodules of M is stationary).

Proof. (iii) $\Rightarrow$ (ii) Let $\mathcal{X}$ be a non-empty set of submodules of M. Let $N_0 \in \mathcal{X}$; if N_0 is not maximal, let $N_1 \supset N_0$ be an element of $\mathcal{X}$ with $N_1 \neq N_0$. If N_1 is not maximal, let $N_2 \supset N_1$ be $\neq N_1$ ...
We construct this way by induction

$$N_0 \subset N_1 \subset \dots \subset N_n \subset \dots$$

which by (iii) stops, so that some N_{n_0} is maximal.

(ii) $\Rightarrow$ (iii) Let $\mathcal{X} = \{ N_n \mid n \geq 0 \}$; this is not empty, so there is some maximal element N_{n_0}; then for $n \geq n_0$, we get

$$N_{n_0} \subset N_n, \quad \text{hence} \quad N_{n_0} = N_n$$

by maximality.

(<u>Note:</u> we have proved that (ii) $\iff$ (iii) for <u>any</u> partially-ordered set $(\mathcal{O}, \leq)$, not only for $\mathcal{O} = \{ N \subset M \}$ with inclusion. This will be useful later.)

(i) $\Rightarrow$ (iii): Let $N_0 \subset N_1 \subset \dots \subset N_n \subset$ be submodules of M; let

$$N' = \bigcup_{n \geq 0} N_n$$

Then $N' \subset M$ is a submodule. Let $(m_1, \text{---}, m_r)$ be a generating set; each m_i belongs to some N_{n_i}, and then

$$N' \subset N_{\max(n_i)}$$

so that $N_{\max(n_i)} \subset N_n \subset N' = N_{\max(n_i)}$ for all $n \geq \max(n_i)$.

(iii) $\Rightarrow$ (i): Let $N \subset M$ be a sub-module, and assume it is <u>not</u> finitely-generated. Let $m_1 \in N - \{0\}$ (which must exist); by induction, let

$$m_{n+1} \in N - \left(\sum_{i=1}^{n} R m_i \right)$$

which again exists. Then let

$$N_n = \sum_{i=1}^{n} R m_i$$

We get $N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_n \subsetneq N_{n+1} \subsetneq \cdots$

so that (iii) fails.

$\square$

The first important property of noetherian modules is that it has good "stability" properties.

<u>Proposition</u>. R ring, M R-module

If $N \subset M$ is a submodule, then

$$M \text{ noetherian}$$
$$\Updownarrow$$
$$N \text{ noetherian and } M/N \text{ noetherian.}$$

<u>Note</u>: as we will see, many constructions

of modules are done by "combining" a submodule and a quotient (see the later discussion of "exact sequences"). So this proposition is very important; notice how it would be false for the condition "finitely-generated" instead of "noetherian".

Before the proof, we state an important corollary.

<u>Corollary</u>. R noetherian ring. Any finitely-generated R-module M is noe-therian.

<u>Proof of Cor.</u> To be finitely generated means that there exists an integer $n \geqslant 0$ and a surjective R-linear map

$$R^n \xrightarrow{f} M.$$

We therefore have $M \simeq R^n / \ker(f)$, and it suffices to show that R^n is a noetherian R-module. This follows by induction on n, using the assumption for $n=1$ and

$$N = R^{n-1} \subset R^n = M$$
$$(x_1, \dots, x_{n-1}) \longmapsto (x_1, \dots, x_{n-1}, 0)$$

with quotient $M/N \simeq R$.
$$(x_i) \longmapsto x_n$$

$\square$

Now we prove the proposition.

<u>Proof.</u> $\underline{\Downarrow}$: If M is noetherian then

(1) $N \subset M$ is (because a submodule of N is one of M)

(2) M/N is (if $I \subset M/N$ then its inverse image by $M \xrightarrow{\pi} M/N$ is finitely-generated, and then $I = \pi(\pi^{-1}(I))$ is also finitely generated [if (m_i) generate $\pi^{-1}(I)$ then $(\pi(m_i))$ generate $\pi(\pi^{-1}(I))$.]

$\underline{\Uparrow}$: suppose that N and M/N are noetherian. Let

$$N_0 \subset N_1 \subset \dots \subset N_n \subset \dots \subset M$$

be a chain of submodules of M. Then we have chains $N_0 \cap N \subset \dots \subset N_n \cap N \subset \dots \subset N$ and $\pi(N_0) \subset \dots \subset \pi(N_n) \subset \dots \subset M/N$.

Since N and M/N are noetherian, there exists $n_0 \geq 0$ s.t.

$$\left\{ \begin{array}{l} N_n \cap N = N_{n+1} \cap N \\ \pi(N_n) = \pi(N_{n+1}) \end{array} \right. \quad \text{for } n \geq n_0.$$

Let $n \geq n_0$ and $m \in N_{n+1}$; there exists $m' \in N_n$ s.t. $\pi(m) = \pi(m')$. Then

$$m - m' \in N \cap N_{n+1} = N \cap N_n$$

so

$$m = m' + (m - m') \in N_n.$$

This shows that $N_n = N_{n+1}$.

$\square$

2 - Hilbert's Theorem

The most important fact about noetherian rings is the fact that this class is stable under adjoining finitely many variables.

Theorem (Hilbert; Noether)

Let R be a noetherian ring and $n \geq 0$ an integer. The polynomial ring $R[X_1, \_\_, X_n]$ is noetherian, or (equivalently) any finitely-generated R-algebra is noetherian.

Proof. First, we explain why the two facts are equivalent: if A is a finitely-generated R-algebra, then there is an isomorphism

$$R[X_1, \_\_\_, X_n]/I \xrightarrow{\sim} A$$

of R-algebras for some $n \geq 0$ and some ideal I. Then A is noetherian as an $R[X_1, \_\_\_, X_n]$-module, and then it is noetherian as an A-module (because an abelian group $N \subset A$ is an A-submodule if and only if it is an $R[X_1, \_\_, X_n]$-submodule).

Now we come to the actual proof. By induction, we need to show that if R is noetherian, then $R[X]$ is also noetherian. We do this by using a (weak) generalization of euclidean division [which is used to prove that $K[X]$ is a PID if K is a field].

Let $I \subset R[X]$ be an ideal. We need to find a finite generating set of I.

We note that for any integer $d \geq 0$, the subset
$$I_d = \{ f \in I \ / \ \deg(f) \leq d \}$$
is an R-submodule of the submodule
$$\{ f \in R[X] \ / \ \deg(f) \leq d \}$$
which is free of rank $d+1$ (with basis

$1, X, \ldots, X^d$). So I_d is finitely-generated because R is noetherian. The difficulty therefore has to do with polynomials of increasing degree in I. In order to handle these, we look for a way of reducing the degree.

To do this, let

$$L = \{0\} \cup \{r \in R \mid \exists f \in I, \ f = rX^d + r_{d-1}X^{d-1} + \ldots + r_0\}$$

"leading"

This is an ideal in R:

(i) clearly if $r \in I$ and $s \in R$ then $rs \in I$

(ii) if $\begin{cases} f_1 = r_1 X^d + \ldots \\ f_2 = r_2 X^e + \ldots \end{cases}$, with $d \geq e$ [$r_1 + r_2 \neq 0$]

then $f_1 + X^{d-e} f_2 = (r_1 + r_2) X^d + \ldots$

So, since R is noetherian, we can find a finite set of (non-zero) generators $\{r_1, \ldots, r_m\}$

and a corresponding finite set $\{f_1, \dots, f_m\}$ of elements of I with

$$f_i = r_i X^{d_i} + (\text{lower degree terms})$$

Let $J \subset R[X]$ be the ideal generated by the f_i; note $J \subset I$.

Let $d = \max_{1 \le i \le m} d_i$. We now show how to "divide" an element $f \in I$ of degree $> d$ to "reduce" to one in I_d: write

$$f = a_e X^e + a_{e-1} X^{e-1} + \dots \qquad e > d$$

Since $a_e \in L$, we can write

$$a_e = \sum_{i=1}^{m} b_i r_i \ , \qquad b_i \in R$$

and then

$$\underbrace{f - \sum_{i=1}^{m} b_i \underbrace{f_i}_{\in J} X^{e-d_i}}_{\in J} = \left(a_e - \sum b_i r_i\right) X^e + (\text{lower degree})$$

so we have found $f_0 \in J$ s.t.

$$\deg(f - f_0) < e.$$

Repeating with $f - f_0$ instead of f, if it has degree $> e$, we construct inductively a polynomial $f_1 \in J$ s.t.

$$f - f_1 \in I_d.$$

If $(f_{m+1}, \dots, f_q)$ are generators of I_d as R-module, this means that f belongs to the (finitely-generated) ideal I' generated by

$$(f_1, \dots, f_q).$$

Since $I_d \subset I'$ this means that

$$I \subset I'$$

and since the converse holds by definition, we have $I = I'$, hence is finitely generated.

$\square$