Chapter V $\quad$ [ACL 8]

The tensor product

Goal: we define and study one of the most fundamental algebraic constructions, that of the tensor product of modules over a ring. This is not only very important in itself, because of applications, but also as a prototype of a certain type of constructions that occur very frequently in modern algebra.

1. Definition

Let R be a ring. The tensor product is a way to "linearize" the concept of bilinear maps of R-modules, viewing them as linear maps on some other module.

Definition (Bilinear map). Let M, N, P

be R-modules. An R-bilinear map

$$b : M \times N \longrightarrow P$$

is a map (of sets) such that

$$b(rm, n) = r\, b(m,n)$$

$$b(m, rn) = r\, b(m,n)$$

$$b(m_1 + m_2, n) = b(m_1, n) + b(m_2, n)$$

$$b(m, n_1 + n_2) = b(m, n_1) + b(m, n_2)$$

for $r \in R$, m, m_1, m_2 in M, n, n_1, n_2 in N.

<u>Lemma</u>. The set $Bil_R(M, N; P)$ of bilinear maps $M \times N \longrightarrow P$ is an R-module with

$$(r_1 b_1 + r_2 b_2)(m,n) = r_1 b_1(m,n) + r_2\, b_2(m,n).$$

Given an R-module Q and an R-linear map $P \xrightarrow{f} Q$, the map

$$b \longmapsto f \circ b$$

is an R-linear map $Bil_R(M, N; P) \longrightarrow Bil_R(M,N;Q)$.

(In other words: we have a functor $Bil_R(M,N)$

on the category of R-modules.)

Examples : (1) $M = N = P = R$; the multi-
-plication map $R \times R \longrightarrow R$ is R-bilinear.

(2) $M = R$, $P = N$; the scalar multipli-
-cation $R \times N \longrightarrow N$ is R-bilinear.

(3) Let $f : M \longrightarrow \mathrm{Hom}_R(N, P)$ be an R-linear map. Then we get an R-bilinear map $b_f : M \times N \longrightarrow P$ by

$$b_f(m,n) = f(m)(n).$$

In fact, any $b \in \mathrm{Bil}(M, N; P)$ is of this form: given $b : M \times N \longrightarrow P$, define

$$f : M \longrightarrow \mathrm{Hom}_R(N, P)$$

such that $f(m)$ is the linear map

$$n \longmapsto b(m,n).$$

In other words: there is a bijection (indeed an R-linear isomorphism) [similarly: there is an isomorphism $\mathrm{Bil}_R(M,N;P) \longrightarrow \mathrm{Hom}_R(N, \mathrm{Hom}_R(M,P))$]

$$\mathrm{Bil}_R(M,N;P) \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$$

The fundamental statement is that we can view $\mathrm{Bil}_R(M,N;P)$ as $\mathrm{Hom}_R(T,P)$ for some other R-module T [depending on M and N] in a "natural way".

<u>Theorem</u> (Whitney). Let R, M, N be as above. There exists an R-module $M \otimes_R N$ ("M tensor N over R") and a bilinear map $\beta : M \times N \longrightarrow M \otimes_R N$ [$\beta \in \mathrm{Bil}_R(M,N;M\otimes_R N)$] which is "universal", in the sense that for any R-module P and any <u>R-bilinear</u> map $b : M \times N \longrightarrow P$, there is a <u>unique</u> <u>R-linear</u> $f : M \otimes_R N \longrightarrow P$ such that

$$b = f \circ \beta.$$

$$\begin{tikzcd} M \times N \arrow[r, "b"] \arrow[d, "\beta"'] & P \\ M \otimes_R N \arrow[ur, "f"'] & \end{tikzcd}$$

bilinear

linear

The <u>pair</u> $(M \otimes_R N, \beta)$ is <u>unique</u>, in the precise sense that if T and $\beta' : M \times N \longrightarrow T$ satisfy the same condition, then there exists a unique R-linear isomorphism

$$T \xrightarrow{f} M \otimes_R N$$

such that $f \circ \beta' = \beta$.

$$\begin{tikzcd} T \arrow[r, "f"] & M \otimes_R N \\ & M \times N \arrow[ul, "\beta'"] \arrow[u, "\beta"'] \end{tikzcd}$$

(In other words: there is a natural equivalence of functors $(R\text{-mod}) \xrightarrow{\Phi} (R-\text{mod})$ between $F = \mathrm{Bil}_R(M, N)$ and $G = h^{M \otimes_R N}$, given by $\left\{ \begin{array}{ccc} G(P) & \xrightarrow{\Phi(P)} & F(P) \\ f & \longmapsto & f \circ \beta \end{array} \right)$.

<u>Notation</u>: we will use the common (but slightly ambiguous notation)

$$\beta(m, n) = m \otimes n$$

for $(m, n) \in M \times N$.

<u>Proof</u>. ① We start with the "<u>uniqueness</u>", which is again a rerun of Yoneda's Lemma in a special case.

So assume we have two modules T, T' with bilinear maps $M \times N \xrightarrow{\beta} T$, $M \times N \xrightarrow{\beta'} T'$, and their "universal" properties. From the data

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "\beta'"'] & T \arrow[dl, bend right, "g"'] \\ T' \arrow[ur, "f"'] & \end{tikzcd}$$

we get R-linear maps

$$T' \xrightarrow{f} T$$

and

$$T \xrightarrow{g} T'$$

by applying the "universal" property of T' to β and that of T to β'. Now we get

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "\beta"'] & T \\ T \arrow[ur, "f\circ g"'] & \end{tikzcd} \qquad \begin{tikzcd} M \times N \arrow[r, "\beta'"] \arrow[d, "\beta'"'] & T' \\ T' \arrow[ur, "g\circ f"'] & \end{tikzcd}$$

which commute $\left[(f\circ g)\circ\beta = f\circ(g\circ\beta) = f\circ\beta' = \beta.\right]$

But $Id_T \circ \beta = \beta$ also; by the uniqueness property of the <u>linear map</u> corresponding to the bilinear map $\beta \in \mathrm{Bil}_R(M,N;T)$, this means that $f\circ g = \mathrm{Id}_T$; similarly $g\circ f = \mathrm{Id}_{T'}$, so f and g give isomorphisms $T' \underset{g}{\overset{f}{\rightleftarrows}} T$.

2 Now we prove the existence. [This proof is both mostly irrelevant — because we basically never need to know how $M\otimes_R N$ is constructed to use it — and very important — because many other algebraic constructions are based on the same principles.] We construct $M\otimes_R N$ as a quotient of a "huge" free module by a submodule that "impose" the bilinearity. (Abstractly, we combine the hom-properties of free modules with those of quotient modules]

Let $\tilde{T}$ be the free R-module with basis indexed by the elements of $M\times N$; write $[(m,n)]$ for the basis vector corresponding to (m,n).

Let $Z \subset \widetilde{T}$ be the submodule generated by the elements for $r \in R$, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$

$$\begin{cases} [(rm, n)] - r[(m,n)] \\ [(m, rn)] - r[(m,n)] \\ [(m_1+m_2, n)] - [(m_1,n)] - [(m_2,n)] \\ [(m, n_1+n_2)] - [(m,n_1)] - [(m,n_2)] \end{cases}$$

Now define $\quad M \otimes_R N = \widetilde{T}/Z$

and $\quad \beta(m,n) = \pi([(m,n)])$

where $\pi : \widetilde{T} \longrightarrow M \otimes_R N$ is the quotient morphism. This pair $(M \otimes_R N, \beta)$ satisfies the conditions we want. First, β is bilinear (exercise).

Next, let P be an R-module. Then first

$$\begin{array}{c} \mathrm{Hom}_R(M \otimes_R N, P) = \mathrm{Hom}_R(\widetilde{T}/Z, P) \\ \downarrow \wr \\ \{ g : \widetilde{T} \to P \ / \ g(Z) = \{0\} \} \end{array}$$

f
$\downarrow$
$f \circ \pi$

Next, R-linear maps $g : \widetilde{T} \longrightarrow P$ are in bijection with maps <u>of sets</u> $M \times N \xrightarrow{h} P$, with

g corresponding to h if and only if

$$\forall (m,n) \in M \times N, \; g([m,n]) = h(m,n). \qquad (*)$$

Therefore $\mathrm{Hom}_R(M \otimes_R N, P)$ corresponds to certain maps $h : M \times N \longrightarrow P$ by the rule

$$\big(f(\beta(m,n)) = \big) \; f(\pi([m,n])) = h(m,n)$$

i.e. $f \circ \beta = h$. The subset of all maps h that arise is dictated by the condition that the corresponding R-linear map $\tilde{T} \xrightarrow{g} P$ satisfies $Z \subset \mathrm{Ker}(g)$, which translates to

$$\begin{cases} g\big([(rm,n)] - r[(m,n)] \big) = 0 \\ g\big([(m,rn)] - r[(m,n)] \big) = 0 \\ g\big([(m_1+m_2,n)] - [(m_1,n)] - [(m_2,n)] \big) = 0 \\ g\big([(m,n_1+n_2)] - [(m,n_1)] - [(m,n_2)] \big) = 0 \end{cases}$$

or equivalently [by $(*)$ and R-linearity]

$$\begin{cases} h(rm,n) = r\,h(m,n) \; ; \; h(m,rn) = r\,h(m,n) \\ h(m_1+m_2,n) = h(m_1,n) + h(m_2,n) \; ; \; h(m,n_1+n_2) = h(m,n_1) + h(m,n_2). \end{cases}$$

In other words, the R-linear map $f \mapsto f \circ \beta$ gives an isomorphism of R-modules

$$\mathrm{Hom}_R(M \otimes_R N, P) \xrightarrow{\sim} \mathrm{Bil}_R(M, N; P)$$

which concludes the proof.

$\square$

<u>Remark.</u> In fact, this last statement is a bit more precise than saying "for every bilinear form there is a unique linear map such that $b = f \circ \beta$", because it specifies that the bijection that this sentence describes is R-linear.

<u>Notation</u>: we will usually denote

$$\beta(m,n) = m \otimes n \qquad \text{[LaTeX: \otimes]}$$

(although there is some ambiguity in doing so).

[Since β is bilinear, these symbols satisfy the rules

$$\begin{cases} rm \otimes n = r(m \otimes n) = m \otimes rn \\ (m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n \\ m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \end{cases}$$

Remark / warning: one must be careful to remember that, in general, not all elements of $M \otimes_R N$ are "pure tensors" of the form $m \otimes n$: they are instead finite linear combinations

$$\sum_{i=1}^{n} m_i \otimes n_i, \qquad \begin{array}{l} n \geq 0 \text{ integer} \\ m_i \in M,\ n_i \in N \end{array}$$

(Indeed, the construction shows that at least the $m \otimes n$ generate $M \otimes_R N$)

Example. Already we can compute some "trivial" tensor products: it may be that $\mathrm{Bil}_R(M, N; P)$ is zero for all P, in which case $M \otimes_R N = \{0\}$. This can sometimes be deduced directly from this formalism.

For instance, let m, n be coprime integers. Then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \{0\}$: write $1 = am + bn$, then $x \otimes y = (am+bn)(x \otimes y) = am(x \otimes y) + bn(x \otimes y)$

$$= \underbrace{amx}_{=0} \otimes y + x \otimes \underbrace{bny}_{=0} = 0$$

for $(x, y) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

2 - Functoriality of the tensor product

So given two R-modules M and N, we have constructed "a" new R-module $M \otimes_R N$.

Is the construction natural? Maybe not, but the characteristic hom-property certainly is. To make this more precise, we get functoriality properties (which will be crucial in computing examples).

<u>Proposition</u>. Let $M \xrightarrow{u} M'$ and $N \xrightarrow{v} N'$ be R-linear maps. There exists a <u>unique</u> R-linear map

$$M \otimes_R N \xrightarrow{u \otimes v} M' \otimes_R N' \quad \text{such that}$$

$$(u \otimes v)(\underbrace{m \otimes n}_{\text{(in } M \otimes_R N)}) = \underbrace{u(m) \otimes v(n)}_{\text{(in } M' \otimes_R N')}$$

for all $(m, n) \in M \times N$.

<u>Proof</u>. The map

$$b \begin{cases} M \times N \longrightarrow M' \otimes_R N' \\ (m, n) \longmapsto u(m) \otimes v(n) \end{cases}$$

is bilinear (it is $\beta' \circ (u, v)$, where β' is the $(u, v): M \times N \longrightarrow M' \times N'$ is linear and $\beta': M' \times N' \longrightarrow M' \otimes_R N'$)

so by definition of $M \otimes_R N$, there exists a unique R-linear map $f : M \otimes_R N \longrightarrow M' \otimes_R N'$ such that $f(\underbrace{m \otimes n}_{\beta(m,n)}) = b(m,n) = u(m) \otimes v(n)$.

$\square$

(Remark. The notation $u \otimes v$ for this linear map is an example where the $\otimes$ symbol is used in a potentially different way than before.)

Lemma. $\mathrm{Id}_M \otimes \mathrm{Id}_N = \mathrm{Id}_{M \otimes N}$

and $(u' \otimes v') \circ (u \otimes v) = (u' \circ u) \otimes (v' \circ v)$

if $\begin{cases} M \xrightarrow{u} M' \xrightarrow{u'} M'' \\ N \xrightarrow{v} N' \xrightarrow{v'} N'' \end{cases}$ are R-linear.

The proof is elementary, using uniqueness.

In particular, if u, v are isomorphisms, then $u \otimes v$ is an isomorphism with inverse $u^{-1} \otimes v^{-1}$.

Remark. We can then say that the tensor product is a functor from the category $(R\text{-mod})^2$ of pairs (M,N) of R-modules to $(R\text{-mod})$, sending (M,N) to $M\otimes_R N$ and a morphism (u,v) to $u\otimes v$. (This requires fixing also the bilinear form!)

3_ Tensor product and direct sums

We can now start computing. Our first goal is to understand the simplest cases, which mean when we have free modules (for instance, vector spaces over a field).

More generally, we will first show how the tensor product behaves "bilinearly" with respect to direct sums (for M and N).

Given a family $(M_i)_{i\in I}$ of R-modules, we recall that $\bigoplus_{i\in I} M_i$ is an R-module with R-linear maps

$$\varphi_j : M_j \longrightarrow \bigoplus M_i , \qquad \psi_j : \bigoplus M_i \longrightarrow M_j$$
$$x_j \longmapsto (0,\_,0,\underbrace{x_j}_{j\text{-th "place"}},0,\_) \qquad (m_i)_{i\in I} \longmapsto m_j$$

for all j.

Proposition - R ring ; M, N R-modules

(1) For any collection $(M_i)_{i\in I}$ (resp. $(N_i)_{i\in I}$) of R-modules, the R-linear map

$$\bigoplus_{i\in I}(M_i \otimes_R N) \xrightarrow{f} \Big(\bigoplus_{i\in I} M_i\Big)\otimes_R N$$

$$\Big[\text{resp. } \bigoplus(M \otimes_R N_i) \longrightarrow M \otimes_R \Big(\bigoplus_i N_i\Big)\Big]$$

such that $m_j \otimes n \longmapsto (0,\dots 0, m_j, 0,\dots)\otimes n$,

$\varphi_j(m_j) =$ — in $M_j \subset \bigoplus M_i$

$$\big[\text{resp. } m \otimes n_j \longmapsto m \otimes (0,\dots,0,n_j,0,\dots)\big)$$

for any $j \in I$, is an isomorphism of R-modules.

(2) The map $M \otimes_R R \longrightarrow M$

$$\big[\text{resp. } R \otimes_R N \longrightarrow N\big]$$

such that $m \otimes r \longmapsto rm$ $\big[\text{resp. } r\otimes n \mapsto rn\big]$

is an isomorphism of R-modules.

Corollary - If M is free with basis $(m_i)_{i\in I}$ and N is free with basis $(n_j)_{j\in J}$, then $M \otimes_R N$ is free with basis $(m_i \otimes n_j)_{(i,j)\in I\times J}$

In particular if K is a field and V, W are finite-dimensional K-vector spaces, then so is $V \otimes_K W$ with

$$\dim (V \otimes_K W) = (\dim V)(\dim W).$$

<u>Proof of corollary</u>. Let $M_i = Re_i$, $N_j = Rf_j$ so that $M = \bigoplus M_i$, $N = \bigoplus N_j$. By the proposition, we get an isomorphism

$$M \otimes_R N = \left(\bigoplus_i M_i\right) \otimes \left(\bigoplus_j N_j\right)$$

$$\xrightarrow{\sim} \bigoplus_{i,j} M_i \otimes N_j .$$

By the second part, $M_i \otimes N_j$, which is isomorphic to $Re_i \otimes N_j$, is isomorphic to N_j, so is free of rank 1. Since it is generated by $e_i \otimes f_j$, this element is a basis, and the above means therefore that $M \otimes_R N$ is free with basis

$(e_i \otimes f_j)_{(i,j) \in I \times J}$, as claimed.

$\square$

<u>Proof of the proposition.</u> We begin with:

(2) First there is a <u>bilinear map</u>

$$R \times M \longrightarrow M$$

such that $(r, m) \longmapsto rm$, so there is an R-linear map $R \otimes_R M \xrightarrow{u} M$ with $u(r \otimes m) = rm$. Conversely, define

$$v : M \longrightarrow R \otimes_R M$$

by $v(m) = 1 \otimes m \in R \otimes_R M$. This is R-linear (because $\otimes$ is bilinear and one argument is fixed) and we have

$$u \circ v(m) = u(1 \otimes m) = m$$

and $v \circ u(x \otimes m) = v(xm) = 1 \otimes xm = x \otimes m$ which implies $v \circ u = \mathrm{Id}_{R \otimes M}$ since the pure tensors generate $R \otimes_R M$.

Now for (1). First we check that

$$\bigoplus_{i\in I}(M_i \otimes_R N) \xrightarrow{f} \left(\bigoplus_{i\in I} M_i\right)\otimes_R N$$

is well-defined. Indeed, to define a linear map $\bigoplus V_i \xrightarrow{f} W$ we must define linear maps $f_j : V_j \longrightarrow W$ for all j, and then put

$$f\left(\sum r_i v_i\right) = \sum r_i f_i(v_i).$$

For any j, we have the injection

$$\varphi_j : M_j \longrightarrow \bigoplus_i M_i$$

and we define $u_j = \varphi_j \otimes Id_N$ to get

$$f_j : M_j \otimes_R N \longrightarrow \left(\bigoplus_i M_i\right)\otimes_R N$$

as required.

To check that f is an isomorphism, we define an inverse: the map

$$\begin{cases} \bigoplus M_i \times N \longrightarrow \bigoplus_i (M_i \otimes_R N) \\ ((m_i), n) \longmapsto \sum_i m_i \otimes n \end{cases}$$

is well-defined (all but finitely m_i are zero for any given element of $\bigoplus M_i$) and R-bilinear, so it defines an R-linear map

$$g : \left(\bigoplus_{i\in I} M_i\right) \otimes_R N \longrightarrow \bigoplus_{i\in I} (M_i \otimes_R N).$$

Now one checks that f and g are reciprocal isomorphisms; for instance

$$(f\circ g)\left((m_i)\otimes n\right) = f\left(\sum_i m_i \otimes n\right)$$

$$= \sum_i f_i(m_i\otimes n)$$

linearity

$$= \sum_i \varphi_i(m_i)\otimes n$$

bilinearity of $\otimes$

$$= \left(\sum_i \varphi_i(m_i)\right)\otimes n$$

$$= (m_i)\otimes n$$

gives one composition, and

$$(g\circ f)(m_j\otimes n) = g\left(\varphi_j(m_j)\otimes n\right)$$

$$= m_j \otimes n, \quad \text{for } j\in I.$$

Since the elements indicated generate the respective modules, we conclude. $\square$

Example. Let K be a field and take

$$M = N = K^2, \text{ with basis } (e_1, e_2).$$

Then $M \otimes_K N$ is a K-vector space of dimension 4, with basis

$$e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2.$$

Using this basis, we can determine the subset of pure tensors. Let

$$v_1 = x_1 e_1 + x_2 e_2 \in M$$

$$v_2 = y_1 e_1 + y_2 e_2 \in N,$$

with x_i, y_i in K.

Then using bilinearity we find that

$$v_1 \otimes v_2 = z_{11}\, e_1 \otimes e_1 + z_{12}\, e_1 \otimes e_2 + z_{21}\, e_2 \otimes e_1$$

where the coordinates z_{ij} are given by $z_{ij} = x_i y_j$. So a pure tensor is an element of the form

$$v = a\, e_1 \otimes e_1 + b\, e_1 \otimes e_2 + c\, e_2 \otimes e_1 + d\, e_2 \otimes e_2$$

for which there exist x_1, x_2, y_1, y_2

s.t.
$$\begin{cases} a = x_1 y_1 \\ b = x_1 y_2 \\ c = x_2 y_1 \\ d = x_2 y_2 \end{cases}$$

Not all elements of $M \otimes_K N$ are of this form, because these satisfy a (non-linear!) equation, namely

$$\underset{x_1 y_1 x_2 y_2}{\underset{\parallel}{ad}} = \underset{x_1 x_2 y_1 y_2}{\underset{\parallel}{bc}}$$

[conversely, if this relation is satisfied, one can check that the element $v \in M \otimes_K N$ is indeed a pure tensor].

Note that in any case, the representation of an element $v = v_1 \otimes v_2$ as a pure tensor is not unique as one can write, e.g. $v = xv_1 \otimes x^{-1}v_2$ for any $x \in K^\times$.

4 - Commutativity and associativity

The following results are other elementary consequences of the definition and are frequently used.

Proposition. M, N R-modules

There exists a unique isomorphism

$$M \otimes_R N \xrightarrow{u_{M,N}} N \otimes_R M$$

such that $u_{M,N}(m \otimes n) = n \otimes m$ for all (m,n) in $M \times N$; we have $u_{M,N}^{-1} = u_{N,M}$.

Proof. The map $\begin{cases} M \times N \longrightarrow N \otimes_R M \\ (m,n) \longmapsto n \otimes m \end{cases}$ is bilinear, so there exists a unique linear map

$$u_{M,N} : M \otimes_R N \longrightarrow N \otimes_R M$$

s.t. $u_{M,N}(m \otimes n) = n \otimes m$. Then $u_{M,N} \circ u_{N,M} = v$ is linear from $N \otimes_R M$ to itself and we have $\quad v(n\otimes m) = n\otimes m \quad$ for $\quad (m,n)\in M\times N$.

Then $v = \mathrm{Id}_{N\otimes_R M}$ because the pure tensors generate $N\otimes_R M$. Similarly $u_{N,M}\circ u_{M,N} = \mathrm{Id}_{M\otimes_R N}$.

$\square$

We now consider the <u>associativity</u> of the tensor product, namely:

<u>Prop.</u> $\quad M, N, P \quad R$-modules.

There exists a unique isomorphism

$$(M\otimes_R N)\otimes_R P \xrightarrow{v} M\otimes_R (N\otimes_R P)$$

such that

$$v\big((m\otimes n)\otimes p\big) = m\otimes(n\otimes p)$$

for all $\quad (m,n,p)\in M\times N\times P$.

<u>Proof.</u> This is a bit more tricky. For any $p\in P$, we consider the map $f_p \begin{cases} N \longrightarrow N\otimes P \\ n \longmapsto n\otimes p \end{cases}$; it is linear. Let $\quad g_p = 1_M\otimes f_p : M\otimes N \to M\otimes(N\otimes P)$

and let $\quad b(x,p) = g_p(x)$; then

$$b : (M \otimes N) \times P \longrightarrow M \otimes (N \otimes P)$$

is bilinear, and therefore there exists a unique linear map

$$v : (M \otimes N) \otimes P \longrightarrow M \otimes (N \otimes P)$$

s.t. $v((m \otimes n) \otimes p) = b(m \otimes n, p)$

$$= g_p(m \otimes n)$$

$$= m \otimes f_p(n) = m \otimes (n \otimes p).$$

We next construct in the "same" way a linear map

$$w : M \otimes (N \otimes P) \longrightarrow (M \otimes N) \otimes P$$

s.t. $w(m \otimes (n \otimes p)) = (m \otimes n) \otimes p$. Then

$$w \circ v = \mathrm{Id}_{(M \otimes N) \otimes P}, \quad v \circ w = \mathrm{Id}_{M \otimes (N \otimes P)}$$

and we are done.

$\square$

Because of this, we usually drop parentheses in multiple tensor products, writing $M \otimes N \otimes P$ only.

5 - Exactness [ACL 7.1]

We have seen that $M \otimes (N_1 \oplus N_2) \simeq M \otimes N_1 \oplus M \otimes N_2$, and this simplifies the computation of tensor products when one of the modules is a direct sum.

However, in the general case, it is much more frequent to have more complicated situations: a module M is given with a submodule $M' \subset M$ and is "built" out of M' and M/M'. What can be said about $M \otimes N$ in terms of $M' \otimes N$ and $M/M' \otimes N$?

Before we address this question, we introduce a more general formalism for such "constructions".

Def. [Exact sequence] R Ring

Let M', M, M'' be R-modules and

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

linear maps. One says that this is an <u>exact</u>

sequence (of R-modules) if $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

[If we have only $\mathrm{Im}(f) \subset \mathrm{Ker}(g)$, i.e. $g \circ f = 0$, on says that this is a complex.]

If we have a sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n,$$

it is exact if each $M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2}$ is exact for $i \leq n-2$ [it is a complex if they are complexes]

Examples. (1) $\{0\} \xrightarrow{0} M \xrightarrow{g} M''$ is always a complex; it is exact if and only if

$$\mathrm{Im}(0) = \ker(g)$$

i.e. if $\mathrm{Ker}(g) = \{0\}$, i.e if g is injective.

(2) $M' \xrightarrow{f} M \xrightarrow{0} \{0\}$ is always a complex; it is exact if and only if

$$\mathrm{Im}(f) = \mathrm{Ker}(0) = M$$

i.e. if and only if f is surjective.

Def. A <u>short exact sequence</u> is an exact sequence of the form

$$\{0\} \longrightarrow \underbrace{M' \xrightarrow{f} M}_{①} \xrightarrow{g} M'' \longrightarrow \{0\}.$$

This means that

① f is injective (Example (1))

② g is surjective (Example (2))

③ We have $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

This means that M' is isomorphic to the submodule $N = \mathrm{Im}(f)$ of M, and M'', as the image of g, is isomorphic to

$$M/\ker(g) = M/\mathrm{Im}(f) = M/N.$$

Example (1) The standard example of a short exact sequence is therefore

$$\{0\} \longrightarrow N \xrightarrow{i} M \xrightarrow{\pi} M/N \longrightarrow \{0\}$$

where i is the injection of a submodule N in M, and π is the canonical projection.

In fact, in a sense all short exact sequences can be identified with one of this type, but the general setting is more flexible.

(2) A special example is when

$$M = M' \oplus M''.$$

We then have

$$\begin{array}{ccccccccc} \{0\} & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & \{0\} \\ & & x & \longmapsto & (x,0) & & & & \\ & & & & (x,y) & \longmapsto & y & & \end{array}$$

which is exact.

But consider for instance

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 2\mathbb{Z}/4\mathbb{Z} & \xrightarrow{i} & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ & & x & \longmapsto & x & & & & \\ & & & & y & \longmapsto & y \bmod 2 & & \end{array}$$

We do <u>not</u> have

$$\mathbb{Z}/4\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

[e.g. the left-hand side contains two solutions of the equation $2x = 0$, and the right-hand side contains 4.]

So there may be more complicated modules "built" from M' and M'' by a short exact sequence $0 \to M' \to M \to M'' \to 0$ than the direct sum $M' \oplus M''$. (Such an M is called an extension of M'' by M').

We know that if $M = M' \oplus M''$, we get $M \otimes_R N \simeq M' \otimes_R N \oplus M'' \otimes_R N$, in other words a short exact sequence

$$0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0.$$

What about a general short exact sequence?

[ACL B.5.1]

Prop. (right-exactness of the tensor product)

If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact, then the sequence

$$M' \otimes_R N \xrightarrow{f \otimes Id_N} M \otimes_R N \xrightarrow{g \otimes Id_N} M'' \otimes_R N \to 0$$

for any R-module N. But $f \otimes Id_N$ is not always injective.

Proof. We need to check that

① $g \otimes Id_N$ is surjective

② $\mathrm{Im}(f \otimes Id_N) = \mathrm{Ker}(g \otimes Id_N)$

The first part is easy (because $M'' \otimes_R N$ is generated by $m'' \otimes n$ and g surjective means that all these are in $\mathrm{Im}(g \otimes Id_N)$].

The inclusion $\mathrm{Im}(f \otimes Id_N) \subset \mathrm{Ker}(g \otimes Id_N)$ is also easy [because $g \circ f = 0$

$\Rightarrow (g \otimes Id_N) \circ (f \otimes Id_N) = (g \circ f) \otimes Id_N = 0$],

but the converse inclusion is not obvious.

We use an abstract argument [see ACL for a direct proof].

Lemma. A sequence of R-modules

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

is exact if and only if, for any R-module P, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(N'', P) \xrightarrow{h_P(g)} \mathrm{Hom}_R(N, P) \xrightarrow{h_P(f)} \mathrm{Hom}_R(N', P)$$

is exact, where as usual $h_P(f)$ is the map

$h \longmapsto h \circ f$ [resp. $h_P(g) : (h \longmapsto h \circ g)$]

Let's see how this lemma helps. For any P, we have the isomorphism

$$\mathrm{Hom}_R(M'' \otimes_R N, P) \xrightarrow{\sim} \mathrm{Hom}_R(M'', \mathrm{Hom}_R(N, P))$$

where $\quad f \longmapsto \tilde{f} \quad$ such that

$$\tilde{f}(n)(m'') = f(m'' \otimes n),$$

and similarly for the others. So the sequence

$$\mathrm{Hom}_R(M'' \otimes_R N, P) \xrightarrow{h_P(g \otimes Id_N)} \mathrm{Hom}_R(M \otimes_R N, P) \xrightarrow{h_P(f \otimes Id_N)} \mathrm{Hom}_R(M' \otimes N, P) \longrightarrow 0 \quad (*)$$

is isomorphic to

$$\mathrm{Hom}_R(M'', Q) \xrightarrow{u} \mathrm{Hom}_R(M, Q) \xrightarrow{v} \mathrm{Hom}_R(M', Q) \longrightarrow 0$$

where $Q = \mathrm{Hom}_R(N, P)$. Under this isomorphism, we have $u = h_Q(g)$, $v = h_Q(f)$, hence the lemma implies that this last sequence is exact, hence also $(*)$; by the lemma again, the sequence

$$M'' \otimes_R N \xrightarrow{f \otimes \mathrm{Id}_N} M \otimes_R N \xrightarrow{g \otimes \mathrm{Id}_N} M' \otimes_R N \longrightarrow 0$$

is exact.

It remains to check the claim about u and v.

This means consider the diagram

$$\begin{tikzcd}
\tilde{\alpha} \in \mathrm{Hom}(M'' \otimes_R N, P) \arrow[r] \arrow[d, "\wr"'] & \tilde{\beta} \in \mathrm{Hom}(M \otimes_R N, P) \arrow[d, "\wr"] \\
\alpha \in \mathrm{Hom}(M'', Q) \arrow[r] & \beta \in \mathrm{Hom}(M, Q)
\end{tikzcd}$$

and given $\alpha : M'' \longrightarrow Q$, show that $\beta = \alpha \circ g$.

The map $\tilde{\alpha}$ is characterised by $\tilde{\alpha}(m'' \otimes n) = \alpha(m'')(n)$

Then $\tilde{\beta}$ " " " $\tilde{\beta}(m \otimes n) = \tilde{\alpha}((g \otimes \mathrm{Id}_N)(m \otimes n))$

$$= \tilde{\alpha}(g(m) \otimes n)$$
$$= \alpha(g(m))(n).$$

Hence β satisfies $\beta(m)(n) = \tilde{\beta}(m \otimes n)$
$$= \alpha(g(m))(n)$$

for all $m \in M$ and $n \in N$. This means that

$$\beta(m) = \alpha(g(m)) \quad \text{for all } m$$

i.e. $\beta = \alpha \circ g$.

We will see later a counterexample to injectivity. $\square$

Proof of the lemma.

$\Longrightarrow$: $\quad N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$ exact

We need to check that

$$0 \to \mathrm{Hom}(N'', P) \xrightarrow{u} \mathrm{Hom}(N, P) \xrightarrow{v} \mathrm{Hom}(N', P)$$

is also, where $u(h) = h \circ g$, $v(h) = h \circ f$.

(i) If $u(h) = 0$ then $h = 0$ on $\mathrm{Im}(g) = N''$, so $h = 0$; hence u is injective.

(ii) To say that $h : N \longrightarrow P$ is of the form $\tilde{h} \circ g$ means that $h \mid \mathrm{Ker}(g) = 0$, i.e. that $h \mid \mathrm{Im}(f) = 0$, i.e. that $h \circ f = v(h) = 0$.

$\Longleftarrow$: we assume exactness of all sequences

$$0 \to \mathrm{Hom}(N'', P) \xrightarrow{u} \mathrm{Hom}(N, P) \xrightarrow{v} \mathrm{Hom}(N', P)$$

as P varies

(i) Pick $P = N''$, $h = g : N \longrightarrow N''$; then since $g = Id_{N''} \circ g = u(Id_{N''})$, we have $g \in \mathrm{Im}(u)$, hence $g \in \mathrm{Ker}(v)$ i.e. $v(g) = g \circ f = 0$.

(ii) We want to show that g is surjective.

Let $P = N''/\mathrm{Im}(g)$ and $h : N'' \longrightarrow P$ the projection; since $h \circ g = 0$, we have $u(g) = 0$ hence by exactness, we have $h = 0$, which means that $P = \{0\}$, so $N'' = \mathrm{Im}(g)$.

(iii) From (i), we know that $\mathrm{Im}(f) \subset \mathrm{Ker}(g)$, and we need the converse inclusion. Pick

$$P = N/\mathrm{Im}(f)$$

and $h : N \longrightarrow P$; then $h \circ f = 0$ so $h \in \mathrm{Ker}(v) = \mathrm{Im}(u)$. So there exists a map $\tilde{h} : N'' \longrightarrow P$ such that $h = \tilde{h} \circ g$.

Hence $\mathrm{Ker}(g) \subset \mathrm{Ker}(h) = \mathrm{Im}(f)$.

$\square$

Example. Let's see that this can be useful.

Let R be a ring and $I, J \subset R$ two ideals.

<u>Claim</u> — There is an isomorphism

$$R/I \otimes_R R/J \xrightarrow{\sim} R/I+J .$$

[For instance $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.]

To prove the claim, we start from the short exact sequence

$$0 \to I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

of R-modules. Now we tensor with R/J, getting an exact sequence

$$\begin{array}{ccccccc} I \otimes_R R/J & \longrightarrow & R \otimes_R R/J & \longrightarrow & R/I \otimes_R R/J & \longrightarrow & 0 \\ & & r\otimes(s+J) \;\; \downarrow \;\; \wr\vert & & & & \\ I \otimes_R R/J & \xrightarrow{u} & R/J & \xrightarrow{v} & R/I \otimes_R R/J & \longrightarrow & 0 . \\ & & rs + J & & & & \end{array}$$

We have therefore an isomorphism induced by v:

$$R/J \Big/ u(I \otimes_R R/J) \xrightarrow{\sim} R/I \otimes_R R/J .$$

(<u>since</u> $\mathrm{Ker}(v) = \mathrm{Im}(u)$!)

By definition the image of u is the set of expressions of the form

$$u\left(\sum_k \underbrace{i_k}_{\in I} \otimes (\overbrace{r_k}^{\in R} + J)\right) = \left(\sum_k i_k r_k\right) + J \in R/J$$

so $\mathrm{Im}(u)$ is simply $I + J/J \subset R/J$ and the isomorphism above becomes

$$\begin{array}{c} R/J \Big/ (I+J)/J \xrightarrow{\sim} R/I \otimes_R R/J \\ \wr\| \\ R/I+J \end{array}$$

which finishes the proof.

<u>Example</u>. The previous computation leads easily to examples of short exact sequences

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

such that

$$0 \longrightarrow M' \otimes_R N \xrightarrow{f \otimes Id_N} M \otimes_R N \xrightarrow{g \otimes Id_N} M'' \otimes_R N \longrightarrow 0$$

is not exact (i.e, $f \otimes Id_N$ is <u>not</u> injective).

Namely, let $R = \mathbb{Z}$, and consider the exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$x \longmapsto 2x$$

$$y \longmapsto y \bmod 2$$

Now let $N = \mathbb{Z}/2\mathbb{Z}$. Tensoring, we get by the example an exact sequence (of $\mathbb{Z}$-modules)

$$\begin{array}{ccccccc} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ \wr\| & & \wr\| & & \wr\| & & \\ \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\tilde{f}} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\tilde{g}} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

and this cannot be exact ($\tilde{g}$ has to be an isomorphism, so $\mathrm{Im}(\tilde{f}) = \{0\} = \mathrm{Ker}(\tilde{g})$, so $\tilde{f}$ cannot be injective). [In fact, tracing the isomorphisms, we find $\tilde{f}(x) = 2x = 0$, as one could expect from the definition of f.]

6_ Base change

The next topic is one of the really important theoretical applications of the tensor product. It describes one to use it to do the analogue/generalization to any algebra $R \longrightarrow A$, and modules M, N over R with a linear map $f: M \longrightarrow N$, of the frequent trick in linear algebra where a real matrix $(a_{ij})_{1 \leq i,j \leq n}$ is viewed as a complex matrix to study its properties (so from a linear map on $\mathbb{R}^n$ one gets "the same" on $\mathbb{C}^n$).

It could already be clear how to do this for vector spaces, or free-modules more generally, but in general, what does it mean?

The answer is the following.

Proposition. [ACL 8.1.12]

Let R be a ring, let $R \xrightarrow{s} A$ be an R-algebra [in particular, A is an R-module with $r \cdot a = s(r)a$, as usual]. Let M be an R-module. Then the R-module

$$M_A = M \otimes_R A$$

has a (unique) structure of <u>A-module</u> such that

$$a \cdot (m \otimes b) = m \otimes ab$$

for all a, b in A and $m \in M$.

<u>Definition</u>. (Base change)

The module M_A is called the "base-change" of M to A.

<u>Proof of the Prop.</u> — First fix $a \in A$. Then the map

$$\begin{cases} M \times A \longrightarrow M \otimes_R A \\ (m, b) \longmapsto m \otimes ab \end{cases}$$

is R-bilinear, hence there is a unique R-linear map

$$\lambda_a : M \otimes_R A \longrightarrow M \otimes_R A$$

s.t. $\lambda_a(m \otimes b) = m \otimes ab$ for $b \in A$, $m \in M$.

The uniqueness implies that

$$\underbrace{\lambda_{a+a'} = \lambda_a + \lambda_{a'}}, \quad \overbrace{\lambda_{aa'} = \lambda_a \circ \lambda_{a'}}^{\text{similar}}$$

(because both are R-linear and send $m \otimes b$ to $m \otimes ab + m \otimes a'b$, by bilinearity of $\otimes$).

It follows that the scalar multiplication

$$a \cdot x = \lambda_a(x)$$

defines an A-module structure on M_A.

$\square$

<u>Remark.</u> Any A-module is also an R-module; the definition shows that M_A gives this way the usual R-module structure on the tensor product $M \otimes_R A$.

<u>Example.</u> If M is free over R with basis (e_i), then M_A is free over A with basis $(e_i \otimes 1_A)$. Indeed, we have $M = \bigoplus R e_i$ so $M_A \xrightarrow{\sim} \bigoplus_{i \in I} (Re_i \otimes_R A)$, and $Re_i \otimes A$ is isomorphic to $R \otimes_R A \simeq A$, which is free of rank 1, and the basis 1_A of A corresponds to $1_R \otimes 1_A$ and to $e_i \otimes 1_A$.

[In particular, if M is an $\mathbb{R}$-vector space of dimension d, then $M_{\mathbb{C}}$ is a $\mathbb{C}$-vector space of the same dimension].

The key point is however (as usual!) that this construction is functorial.

Proposition. Let $R \xrightarrow{s} A$ be an R-algebra, let $M \xrightarrow{f} N$ be an R-linear map. Then the map $f_A = f \otimes \mathrm{Id}_A : M_A \longrightarrow N_A$ is A-linear for the A-module structures defined above.

R-linear

Moreover, $(\mathrm{Id}_M)_A = \mathrm{Id}_{M_A}$ and if $N \xrightarrow{g} P$ is R-linear, then $(g \circ f)_A = g_A \circ f_A : M \longrightarrow P$.

Proof. Since $f \otimes \mathrm{Id}_A$ is R-linear, it is $\mathbb{Z}$-linear (additive), which means that we only need to check that $f_A(ax) = a f_A(x)$ to conclude that f_A is A-linear.

Let $\lambda_a : x \longmapsto ax$ be the multiplication by A. It is R-linear, and we need to check that $f_A \circ \lambda_a = \lambda_a \circ f_A$. Both are linear maps $M \otimes_R A \longrightarrow N \otimes_R A$ such that

$$m \otimes b \longmapsto f(m) \otimes ab$$

$[(f_A \circ \lambda_a)(m \otimes b) = f_A(m \otimes ab) = f(m) \otimes ab$, and
$(\lambda_a \circ f_A)(m \otimes b) = \lambda_a(f(m) \otimes b) = f(m) \otimes ab]$.

so they coincide.

The composition formulas are then easy.

$\square$

Example. (1) Let K be a field and M, N finite-dimensional K-vector spaces. Let $f : M \to N$ be K-linear. Choose bases (x_i) and (y_j) of M and N, and let (a_{ij}) be the matrix of f with respect to these.

Claim. For a field extension $K \subset L$

[so L is a K-algebra], the matrix of f_L with respect to $(x_i \otimes 1_L)$, $(y_j \otimes 1_L)$, is also (a_{ij}) [viewed as having $a_{ij} \in L$].

Indeed, by definition we have

$$f(x_k) = \sum_j a_{jk} y_j, \quad 1 \le k \le \dim(M)$$

so

$$f_A(x_k \otimes 1) = f(x_k) \otimes 1 = \left(\sum a_{jk} y_j\right) \otimes 1$$
$$= \sum a_{jk} \, (y_j \otimes 1)$$

which gives the result.

(2) Here is an example of application of base change. Consider two elements α, β of $\mathbb{C}$ which are algebraic over $\mathbb{Q}$, i.e., there are polynomials p_α, p_β in $\mathbb{Q}[X]$, non-zero, such that

$$p_\alpha(\alpha) = 0, \quad p_\beta(\beta) = 0.$$

One knows (from the theory of fields) that $\alpha+\beta$, $\alpha\beta$ are also algebraic. Can one find an <u>explicit</u>

polynomial equation which they satisfy?

Here is one way:

Step 1. Find matrices $A_\alpha \in M_n(\mathbb{Q})$ and $A_\beta \in M_m(\mathbb{Q})$ (with $n = \deg(p_\alpha)$, $m = \deg(p_\beta)$) with characteristic polynomial p_α and p_β, respectively.

Step 2. Let u_α, u_β be the linear maps $\mathbb{Q}^n \longrightarrow \mathbb{Q}^n$ or $\mathbb{Q}^m \longrightarrow \mathbb{Q}^m$ associated to A_α, A_β.

Let $A_{\alpha\beta}$ = matrix of $u_\alpha \otimes u_\beta$ w.r.t the basis $(\underbrace{e_i}\otimes \underbrace{f_j})_{\substack{1\le i\le n\\ 1\le j\le m}}$ of $\mathbb{Q}^n\otimes\mathbb{Q}^m$.

canonical bases of $\mathbb{Q}^n$ and $\mathbb{Q}^m$

$A_{\alpha+\beta}$ = matrix of $u_\alpha \otimes \mathrm{Id}_{\mathbb{Q}^m} + \mathrm{Id}_{\mathbb{Q}^n} \otimes u_\beta$ w.r.t. $(e_i \otimes f_j)$.

Then the characteristic polynomials $p_{\alpha\beta}$ of $A_{\alpha\beta}$ and $p_{\alpha+\beta}$ of $A_{\alpha+\beta}$ satisfy

$$p_{\alpha\beta}(\alpha\beta) = 0, \qquad p_{\alpha+\beta}(\alpha+\beta) = 0.$$

Why is that? Let $x_\alpha \in \mathbb{C}^n = \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{C}$

$x_\beta \in \mathbb{C}^m = \mathbb{Q}^m \otimes_{\mathbb{Q}} \mathbb{C}$

be eigenvectors for α and β: $A_\alpha x_\alpha = \alpha x_\alpha$ when we view A_α as a matrix with coefficients in $\mathbb{C}$, and similarly $A_\beta x_\beta = \beta x_\beta$. This translates to

$$\begin{cases} (u_\alpha \otimes Id_{\mathbb{C}})\, x_\alpha = \alpha\, x_\alpha \\ (u_\beta \otimes Id_{\mathbb{C}})\, x_\beta = \beta\, x_\beta \end{cases}$$

[here $u_\alpha \otimes Id_{\mathbb{C}} : \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{C}$].

Now let $x_{\alpha\beta} = x_\alpha \otimes x_\beta \in \mathbb{C}^n \otimes \mathbb{C}^m$.

Then

$$\begin{aligned}(u_{\alpha\beta} \otimes Id_{\mathbb{C}})(x_{\alpha\beta}) &= (u_\alpha \otimes Id_{\mathbb{C}}) \otimes (u_\beta \otimes Id_{\mathbb{C}}) (x_\alpha \otimes x_\beta) \\ &= u_\alpha(x_\alpha) \otimes u_\beta(x_\beta) \\ &= \alpha\beta\,(x_\alpha \otimes x_\beta).\end{aligned}$$

Moreover $x_{\alpha\beta} \neq 0$ [because it can be part of a basis of $\mathbb{C}^n \otimes \mathbb{C}^m$].

Similarly, we get

$$u_{\alpha+\beta}(x_{\alpha+\beta}) = \left((u_\alpha \otimes Id_{\mathbb{C}}) \otimes (Id_{\mathbb{Q}^m} \otimes Id_{\mathbb{C}})\right)(x_\alpha \otimes x_\beta)$$

$$+ \left((Id_{\mathbb{Q}^n} \otimes Id_{\mathbb{C}}) \otimes (u_\beta \otimes Id_{\mathbb{C}})\right)(x_\alpha \otimes x_\beta)$$

$$= u_\alpha(x_\alpha) \otimes x_\beta + x_\alpha \otimes u_\beta(x_\beta)$$

$$= (\alpha+\beta)\,(x_\alpha \otimes x_\beta).$$

7. Examples of base change

We describe here some especially useful and important examples of base change

(1) Quotients

Proposition. Let R be a ring, $I \subset R$ an ideal. For any R-module M, there is an isomorphism $M \otimes R/I \xrightarrow{u} M/IM$ of R/I-modules such that $u(m \otimes (r+I)) = rm + IM$.

Proof. First there is an R-bilinear map

$$M \times R/I \longrightarrow M/IM$$

sending $(m, r+I)$ to the class of rm modulo IM [if r is replaced by $r+i$, then rm is replaced by $rm+im$, and $im \in IM$].

This defines $u : M \otimes_R R/I \longrightarrow M/IM$ as an R-linear map. Both sides are R/I-modules, and u is R/I-linear.

$$\Big[\ u\big((s+I)\,(m\otimes(r+I))\big) = u\big(m\otimes(rs+I)\big) = rsm + IM$$

$$= (s+I)\,(rm\ IM).\ \Big]$$

It suffices then to construct the inverse: we define first $\tilde{v} : M \longrightarrow M\otimes_R R/I$ R-linear by $\tilde{v}(m) = m\otimes 1_{R/I}$. Then since

$$\tilde{v}(im) = im\otimes 1_{R/I} = i\,(m\otimes 1_{R/I})$$

$$= m\otimes(i \bmod I) = 0,$$

we get an induced R-linear map

$$v : M/IM \longrightarrow M\otimes_R R/I$$

with $v(m+IM) = m \otimes 1_{R/I}$. Then check that v is R/I-linear, and then

$$v \circ u(m \otimes (r+I)) = v(rm + IM)$$

$$= rm \otimes 1_{R/I} = m \otimes (r+I)$$

and $u \circ v\,(m+IM) = u(m \otimes 1_{R/I}) = m + IM$

imply the result.

$\square$

(2) Localization

Proposition. R ring, $S \subset R$ multiplicative.

For any R-module M, there is an $S^{-1}R$-linear isomorphism

$$M \otimes S^{-1}R \xrightarrow{u} S^{-1}M$$

where the right-hand side is $S \times M/\sim$, where

$$\frac{m}{s} = \frac{m'}{s'} \iff \exists t \in S,\quad t(s'm - sm) = 0,$$

and $\frac{r}{s} \cdot \frac{m}{s'} = \frac{rm}{ss'}$, $\frac{m}{s} + \frac{m'}{s'} = \frac{s'm + sm'}{ss'}$.

This isomorphism is characterized by $u(m \otimes \frac{r}{s}) = \frac{rm}{s}$.

Proof. First, one checks that $S^{-1}M$ is indeed well-defined and is an $S^{-1}R$-module. Then

$$b \begin{cases} M \times S^{-1}R \longrightarrow S^{-1}M \\ \left(m, \frac{r}{s}\right) \longmapsto \frac{rm}{s} \end{cases}$$

is well-defined: if $\frac{r}{s} = \frac{r'}{s'}$, so

$\exists\, t \in S,\quad t(s'r - sr') = 0$

then $\quad t(s'rm - sr'm) = 0 \implies \frac{rm}{s} = \frac{rm'}{s'}$.

The map b is R-bilinear, hence an R-linear map $\quad u : \quad M \otimes_R S^{-1}R \longrightarrow S^{-1}M$ with

$u\left(m \otimes \frac{r}{s}\right) = \frac{rm}{s}$.

In fact, u is $S^{-1}R$-linear:

$$u\left(\frac{r'}{s'}\left(m \otimes \frac{r}{s}\right)\right) = u\left(m \otimes \frac{rr'}{ss'}\right)$$

$$= \frac{rr'm}{ss'} = \frac{r'}{s'} \cdot \left(\frac{rm}{s}\right).$$

Now we construct the inverse: we start with

$$\tilde{v} : \quad M \longrightarrow M \otimes S^{-1}R$$
$$m \longmapsto m \otimes 1$$

which is R-linear. Since the target is an

$S^{-1}R$-module, $\tilde{v}$ "extends" to $S^{-1}M$ by

$$v\left(\frac{m}{s}\right) = \frac{1}{s}\,\tilde{v}(m) = m \otimes \frac{1}{s}.$$

[this holds for <u>any</u> $S^{-1}R$-module N and R-linear map $f: M \longrightarrow N$]

Then note that

$$u \circ v\left(\frac{m}{s}\right) = u\left(m \otimes \frac{1}{s}\right) = \frac{m}{s}$$

$$v \circ u\left(m \otimes \frac{r}{s}\right) = v\left(\frac{rm}{s}\right) = rm \otimes \frac{1}{s}$$

$$= m \otimes \frac{r}{s}.$$

$\square$

[ACL 7.5.12]

<u>Corollary</u>. - If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact, then

$$0 \to M' \otimes_R S^{-1}R \xrightarrow{f_S} M \otimes_R S^{-1}R \xrightarrow{g_S} M'' \otimes_R S^{-1}R \to 0$$

is exact.

<u>Proof</u>. We only need to show that $f_S = f \otimes Id_{S^{-1}R}$ is injective. If we identify $M' \otimes S^{-1}R$ and $M \otimes S^{-1}R$ with $S^{-1}M'$ and $S^{-1}M$ then f_S

is defined by $f_S\left(\frac{m'}{s}\right) = (f \otimes Id)\left(m' \otimes \frac{1}{s}\right)$

$= f(m') \otimes \frac{1}{s} = \frac{f(m')}{s}.$

Let $\frac{m'}{s} \in Ker(f_S)$; then there exists $t \in S$ such that $t f(m') = 0$ so $tm' \in Ker(f)$

so $tm' = 0$ so $\frac{m'}{s} = 0$ in $S^{-1}M'$.

$\square$

(3) Principal Ideal Domains

Consider a PID R and its fraction field K.

Prop. Let M be a finitely-generated R-module. Then M_K is a vector space over K of dimension equal to the rank $n \geq 0$ of the free part of M: $M \simeq R^n \oplus T$, where T is a torsion R-module.

In other words: $R \otimes_R K \simeq K$

and $R/I \otimes_R K = \{0\}$

for any ideal $I \neq \{0\}$ in R.

[This holds for any integral domain]

Proof. Only the second requires a proof. But if $i \in I - \{0\}$, then
$$(r+I) \otimes 1_K = (r+I) \otimes \frac{i}{i} = (ir+I) \otimes \frac{1}{i} = 0$$
for any $r \in R$, so $R/I \otimes_R K = 0$.

$\square$

8. Tensor product of algebras

Proposition. Let R be a ring and
$$s_A : R \longrightarrow A, \quad s_B : R \longrightarrow B$$
be R-algebras. Then $C = A \otimes_R B$ has a structure of R-algebra with
$$(a_1 \otimes b_1) \cdot_C (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$
and $r \cdot (a \otimes b) = ra \otimes b = a \otimes rb$

It satisfies: for any R-algebra D,

with structure morphism $s_D : A \longrightarrow D$

$$\mathrm{Hom}_{(R\text{-alg})}(A \otimes_R B, D) \xrightarrow[\text{[bijection]}]{\varphi} \mathrm{Hom}_{(R\text{-alg})}(A, D) \times \mathrm{Hom}_{(R\text{-alg})}(B, D)$$

```latex
\begin{tikzcd}
A \arrow[r, "f"] & D \\
R \arrow[u, "s_A"] \arrow[ur, "s_D"] \arrow[r, "s_B"'] & B \arrow[u, "g"']
\end{tikzcd}
```

(note $f \circ s_A = g \circ s_B = s_D$)

where $\sigma : A \otimes_R B \longrightarrow D$ maps to (f, g) where

$$f(a) = \sigma(a \otimes 1_B), \quad g(b) = \sigma(1_A \otimes b).$$

Proof. First, we have an R-linear map

$$C \otimes_R C \xrightarrow{m} C$$

$$C \otimes_R C = (A \otimes B) \otimes (A \otimes B) \cong (A \otimes A) \otimes (B \otimes B) \xrightarrow{m_A \otimes m_B} C$$

where m_A, m_B correspond to the R-bilinear multiplications $m_A : A \times A \longrightarrow A$, $m_B : B \times B \longrightarrow B$.

Then
$$m((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) = (m_A \otimes m_B)((a_1 \otimes a_2) \otimes (b_1 \otimes b_2))$$
$$= a_1 a_2 \otimes b_1 b_2$$

So we have an R-bilinear

$$m_C \begin{cases} C \times C \longrightarrow C \\ (c_1, c_2) \longmapsto m(c_1 \otimes c_2) \end{cases}$$

We claim first that C with m_C as product and with $1_C = 1_A \otimes 1_B \in C$ is a ring (commutative with unit). For instance:

(1) m_C is <u>commutative</u> [it suffices to check that $m_C(a_1 \otimes b_1, a_2 \otimes b_2) = m_C(a_2 \otimes b_2, a_1 \otimes b_1)$, and both are equal to $a_1 a_2 \otimes b_1 b_2$].

(2) m_C is <u>associative</u> [again by linearity one reduces to $c_1(c_2 c_3) = (c_1 c_2) c_3$ for $c_i = a_i \otimes b_i$, and both are $a_1 a_2 a_3 \otimes b_1 b_2 b_3$].

Now since C is an R-module, it is an R-algebra with the scalar multiplication determined by $r \cdot (a \otimes b) = ra \otimes b = a \otimes rb$.

Finally, we prove the last statement.

Note that given $\sigma : A \otimes_R B \longrightarrow D$, which is a morphism of R-algebras, and we define

$$f(a) = \sigma(a \otimes 1_B), \quad g(b) = \sigma(1_A \otimes b)$$

$$\begin{tikzcd} A \arrow[r, "f"] & D \\ R \arrow[u, "s_A"] \arrow[ur, "s_D"] \arrow[r, "s_B"'] & B \arrow[u, "g"'] \end{tikzcd}$$

Then we get

$$g(s_B(\lambda)) = \sigma(1_A \otimes \lambda) = s_D(\lambda)$$

so g is an R-algebra morphism, and similarly for f, so the map $\sigma \mapsto (f,g)$ is well-defined. We can construct the reciprocal bijection as follows: given (f,g) as above, there is an R-bilinear map

$$\beta \left\{ \begin{array}{ccc} A \times B & \xrightarrow{b} & D \\ (a,b) & \longmapsto & f(a)\,g(b) \end{array} \right.$$

and therefore an R-linear morphism

$$\sigma \left\{ \begin{array}{ccc} A \otimes_R B & \longrightarrow & D \\ a \otimes b & \longmapsto & f(a)\,g(b) \end{array} \right.$$

We claim that σ is an R-algebra morphism: for any $c \in C$, with $\lambda_c : \begin{array}{ccc} A \otimes_R B & \longrightarrow & A \otimes_R B \\ x & \longmapsto & cx \end{array}$

this means that $\quad \mu_d : \begin{array}{ccc} D & \longrightarrow & D \\ y & \longmapsto & dy \end{array}$

$$\circledast \qquad \sigma \circ \lambda_c = \mu_{\sigma(c)} \circ \sigma$$

It suffices to check this for $c = a_0 \otimes b_0$ [because the set of c's for which $\circledast$ holds is an R-submodule] in which case we can also restrict to checking for $a \otimes b$, and then

$$(\sigma \circ \lambda_c)(a \otimes b) = \sigma(a_0 a \otimes b_0 b)$$
$$= f(a_0 a)\, g(b_0 b)$$
$$= (f(a_0) g(b_0))\, f(a) g(b)$$
$$= \mu_{\sigma(a_0 \otimes b_0)}(\sigma(a \otimes b))$$

gives the result.

It is also an R-algebra morphism: we have

$$\sigma(r \cdot (a \otimes b)) = \sigma(ra \otimes b) = f(ra)\, g(b)$$
$$= r\, \sigma(a \otimes b).$$

We can therefore define $\psi : (f, g) \longmapsto \sigma$.

Then φ and ψ are reciprocal isomorphisms:

① $(\varphi \circ \psi)(f, g)$ is the pair (f_1, g_1) such that $f_1(a) = \sigma(a \otimes 1_B) = f(a)$ since

$\sigma(a \otimes 1_B) = f(a)\, g(1_B) = f(a)$, and similarly

$g_1 = g$.

(2) $(\psi \circ \varphi)(\sigma)$ is the morphism $A \otimes_R B \xrightarrow{\sigma_1} D$

such that $a \otimes b \longmapsto f(a)\, g(b)$ where

$$f(a) = \sigma(a \otimes 1_B), \quad g(b) = \sigma(1_A \otimes b)$$

so $f(a)\, g(b) = \sigma((a \otimes 1_B)(1_A \otimes b)) = \sigma(a \otimes b)$,

hence $\sigma_1 = \sigma$.

$\square$

<u>Example</u> - (1) Let R be any ring. There

is a canonical isomorphism

$$\begin{cases} R \otimes_{\mathbb{Z}} \mathbb{Z}[X] \xrightarrow{\sim} R[X] \\ r \otimes p \longmapsto rp \end{cases}$$

of $\mathbb{Z}$-algebras (i.e. of rings).

Similarly, there is a canonical isomorphism

$$R[X] \otimes_R R[Y] \xrightarrow{\sim} R[X, Y]$$

where $$p(X) \otimes q(Y) \longmapsto p(X)\, q(Y).$$

(Indeed, in both cases, we have free modules and the indicated maps are R-linear isomorphisms, so it suffices to check that they are ring or algebra morphisms; this can be checked on pure tensors and is easy in both cases).

(2) Let X be a finite set. Denote by $C(X)$ the $\mathbb{C}$-algebras of functions $f : X \longrightarrow \mathbb{C}$.

Then for finite sets X and Y we have a $\mathbb{C}$-algebra isomorphism
$$C(X) \otimes_{\mathbb{C}} C(Y) \xrightarrow[\sim]{\varphi} C(X \times Y)$$
where $\varphi(f \otimes g)$ is the function $(x, y) \longmapsto f(x)\, g(y)$.

Indeed, it is easy to see that φ exists as a $\mathbb{C}$-linear map; it is surjective because $C(X \times Y)$ is generated by the characteristic functions $\delta_{(x,y)}$ of points $(x,y) \in X \times Y$, and

$\delta_{(x,y)} = \varphi(\delta_x \otimes \delta_y)$. It is therefore an isomorphism because $\dim C(Z) = \mathrm{Card}(Z)$ in general and $\mathrm{Card}(X \times Y) = \mathrm{Card}(X)\,\mathrm{Card}(Y)$

$$= \dim \left(C(X) \otimes_{\mathbb{C}} C(Y) \right).$$

And finally one checks as usual on pure tensors that φ is a ring morphism.

(3) The following example plays an important role in algebraic geometry and illustrates many topics we have considered.

<u>Proposition</u> - Let $f : A \longrightarrow B$ be a ring-morphism. Let $\mathfrak{p} \subset A$ be a prime ideal, and let $k_{\mathfrak{p}} = A_{\mathfrak{p}}/_{\mathfrak{p}A_{\mathfrak{p}}}$ be the residue field of the local ring $A_{\mathfrak{p}}$; it is an A-algebra by the composition

$$A \longrightarrow A_{\mathfrak{p}} \longrightarrow k_{\mathfrak{p}}.$$

Let C be the A-algebra $C = B \otimes_A k_{\mathfrak{p}}$.

There is a bijection

$\{\text{prime ideals in } C\} \xrightarrow{\sim} \{\text{prime ideals } q \subset B \text{ such that } f^{-1}(q) = p\}$

Proof. The idea is to first compute C in a different way.

Claim: There is a unique A-algebra isomorphism $C \xrightarrow{g} S^{-1}B/I$,

where $\begin{cases} S = f(A-p) \subset B \\ I \subset S^{-1}B \text{ is the ideal generated by } f(p) \end{cases}$

such that

$$g\left(b \otimes \left(\frac{1}{s} + pA_p\right)\right) = \frac{f(1)b}{f(s)} + I.$$

We assume first that this is correct and deduce the proposition as follows:

(1) prime ideals in C are in bijection

with prime ideals in $S^{-1}B$ containing I, i.e. prime ideals in $S^{-1}B$ containing $f(p)$

(2) prime ideals in $S^{-1}B$ are in bijection with prime ideals q in B such that

$$q \cap S = q \cap f(A-p) = \emptyset$$

[(1) and (2) have been explained in Chapter 3]

A prime ideal $q \subset B$ satisfying (2) will also satisfy (1) if and only if $q \supset f(p)$.

So we get a bijection

$$\{\text{prime ideals in } C\} \xrightarrow{\sim} \left\{ \begin{array}{l} q \subset B \text{ prime} \\ \text{with } f(p) \subset q \\ \text{and } q \cap f(A-p) = \emptyset \end{array} \right\}$$

However, $q \cap f(A-p) = \emptyset$ and $f(p) \subset q$ means that $f^{-1}(q) = p$, so we are done.

[$(f(p) \subset q) \iff (p \subset f^{-1}(q))$; and $q \cap f(A-p) = \emptyset$ implies $f^{-1}(q) \subset p$, since if $x \notin p$, then $f(x) \notin q$]

It remains to prove the claim. First, g is defined because

$$\beta \left\{ \begin{array}{ccc} B \times k_p & \longrightarrow & S^{-1}B/I \\ (b, \frac{r}{s} + pA_p) & \longmapsto & \frac{b f(r)}{f(s)} + I \end{array} \right.$$

is well-defined and is A-bilinear

[for instance, if $x = \frac{r}{s}$ with $r \in p$, then $\frac{b f(r)}{f(s)} \in I$, so β is well-defined].

Then g is A-linear is easy to check.

Finally we construct an inverse

$$h : S^{-1}B/I \longrightarrow C$$

by

$$h\left(\frac{b}{f(r)} + I\right) = b \otimes \left(\frac{1}{r} + pA_p\right).$$

$\square$

For example, consider $f : \mathbb{Z} \to \mathbb{Z}[i]$ and p prime.

Then the prime ideals $q \subset \mathbb{Z}[i]$ s.t. $q \cap \mathbb{Z} = p\mathbb{Z}$ are in bijection with prime ideals in the ring

$$\mathbb{Z}[i] \otimes \mathbb{Z}/p\mathbb{Z} \simeq \mathbb{Z}[x]/(x^2+1) \otimes \mathbb{Z}/p\mathbb{Z}$$

$$\simeq \mathbb{Z}/p\mathbb{Z}[x]/(x^2+1)$$

[The first isomorphism is induced by $X \longmapsto i$ and the second is given by

$$(f+(X^2+1)) \otimes (a+p\mathbb{Z}) \longmapsto af + (X^2+1)\, \mathbb{Z}/p\mathbb{Z}[X] \quad]$$

The Chinese Remainder Theorem describes

$$\mathbb{Z}/p\mathbb{Z}[X] \Big/ (X^2+1)$$

in terms of the factorization of X^2+1 in $\mathbb{Z}/p\mathbb{Z}[X]$.

Since $x^2+1=0 \bmod p$ is equivalent for p odd to $\begin{cases} x^4 = 1 \\ x^2 \neq 1 \end{cases}$ modulo p

the existence of a solution is equivalent to $p \equiv 1 \bmod 4$.

[because $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic of order $p-1$ and must contain an element of order exactly 4]

So there are two ideals q if $p \equiv 1 \bmod 4$, and a single one if $p \equiv 3 \bmod 4$.

(One can check that $\mathbb{Z}[i]$ is principal; if $a+ib$ satisfies $(a+ib)\mathbb{Z}[i] \cap \mathbb{Z} = p\mathbb{Z}$, then one can also show that $a^2+b^2=p$; the existence of such a, b for $p \equiv 1 \bmod 4$ is a theorem of Fermat.)

9 – Multilinear algebra [mostly without proofs]

One can generalize bilinear maps to multilinear maps, and then ask for a similar "linearization" procedure. But it turns out that the tensor product can be used for all of these by iterating.

Definition – $k \geq 1$ integer

(1) $M_1, \ldots, M_k, N$ R-modules

A map $\mu : M_1 \times \cdots \times M_k \longrightarrow N$ is R-multilinear if each

$$\left\{ \begin{array}{l} m \longmapsto \mu(m_1, \ldots, m_{i-1}, m, m_{i+1}, \ldots) \\ M_i \longrightarrow N \end{array} \right.$$

is linear. The set of such μ is an R-module denoted $\mathrm{Mult}_R(M_1, \ldots, M_k ; N)$.

(2) If $M_i = M$ for all i, then a μ as above is

a) <u>symmetric</u> if

$$\mu(m_{\sigma(1)}, \dots, m_{\sigma(k)}) = \mu(m_1, \_\_\_, m_k)$$

for all permutations σ

b) <u>alternating</u> if

$$\mu((m_i)) = 0 \quad \text{if there are } i_1 \neq i_2$$

$$\text{s.t. } m_{i_1} = m_{i_2}.$$

Denote $\mathrm{Sym}^k_R(M, N)$ and $\mathrm{Alt}^k_R(M, N)$ the corresponding submodules of $\mathrm{Mult}_R(\underbrace{M, \dots, M}_{k \text{ times}}; N)$.

<u>Proposition.</u> $k \geq 2$; $M_1, \_\_\_, M_k$ R-modules.

For any R-module N, there is an isomorphism of R-modules

$$\mathrm{Hom}_R(M_1 \otimes_R \cdots \otimes_R M_k, N) \xrightarrow{\sim} \mathrm{Mult}_R(M_1, \dots, M_k; N)$$

such that $f : M_1 \otimes_R \cdots \otimes_R M_k \longrightarrow N$ is mapped to the map μ such that

$$\mu(m_1, \dots, m_k) = f(m_1 \otimes \cdots \otimes m_k).$$

<u>Proof.</u> Left as an exercise!

Consider $M_i = M$ for all i.

Let $S \subset M \otimes_R \text{———} \otimes_R M \; (= M^{\otimes k})$ be the submodule generated by elements of the form

$$m_1 \otimes \cdots \otimes m_k - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)} .$$

Then by properties of quotients, we have an iso-morphism

$$\mathrm{Hom}_R (M^{\otimes k}/_S , N) \xrightarrow{\sim} \mathrm{Sym}^k_R (M, N).$$

Similarly with $A \subset M^{\otimes n}$ generated by all

$$m_1 \otimes \cdots \otimes m_{i-1} \otimes m \otimes m_{i+1} \otimes \cdots \otimes m_{j-1} \otimes m \otimes m_{j+1} \otimes \cdots$$

we obtain

$$\mathrm{Hom}_R (M^{\otimes k}/_A , N) \xrightarrow{\sim} \mathrm{Alt}^k_R (M, N).$$

Def. $\mathrm{Sym}^k (M) = M^{\otimes k}/_S$

$\bigwedge^k (M) = M^{\otimes k}/_A$

are called the k-th symmetric power and the k-th exterior power of M respectively.

Example - (1) Suppose M_i is free of rank d_i for $1 \le i \le k$, with basis $(e_{i,j})_{1 \le j \le d_i}$. Then by induction from the case $k=2$, we see that $M_1 \otimes \cdots \otimes M_k$ is free of rank $d_1 \cdots d_k$ with basis $(e_{1,j_1} \otimes \cdots \otimes e_{k,j_k})$.

(2) Let M be free of rank d, with basis $(e_i)_{1 \le i \le n}$

Then: a) $\mathrm{Sym}^k(M)$ is free of rank

$$\binom{n+k-1}{k}$$

with a basis given by the classes of

$$x_1 \otimes \cdots \otimes x_k$$

where $x_i \in \{e_1, \ldots, e_n\}$ for all n. [The number of these is the number of monomials in n variables with degree k].

b) $\bigwedge^k M$ is free of rank

$$\binom{n}{k}$$

with a basis given by the classes of

$$e_{i_1} \otimes \cdots \otimes e_{i_k}$$

for $1 \leq i_1 < \cdots < i_k \leq n$.

The class of an element $m_1 \otimes \cdots \otimes m_k$ is denoted

"wedge" $m_1 \wedge \cdots \wedge m_k \in \bigwedge^k M$.

In particular $\bigwedge^k M = \{0\}$ if $k \geq n+1$ and

$\bigwedge^n M$ is free of rank 1, with basis

$$e_1 \wedge \cdots \wedge e_n.$$

Proposition. $\mathrm{Sym}^k M$ and $\bigwedge^k M$ are functorial:

given $f : M \longrightarrow N$ linear, there are associated

linear maps $\mathrm{Sym}^k M \xrightarrow{\mathrm{Sym}^k(f)} \mathrm{Sym}^k N$ and

$$\bigwedge^k M \xrightarrow{\bigwedge^k f} \bigwedge^k N$$

characterized by

$$\left\{\begin{matrix}\mathrm{Sym}^k f \\ \bigwedge^k f\end{matrix}\right\}(\text{class of } x_1 \otimes \cdots \otimes x_k)$$

$$= \text{class of } f(x_1) \otimes \cdots \otimes f(x_k)$$

We have $\mathrm{Sym}^k \mathrm{Id} = \mathrm{Id}$, $\bigwedge^k \mathrm{Id} = \mathrm{Id}$

and $\mathrm{Sym}^k(f \circ g) = \mathrm{Sym}^k f \circ \mathrm{Sym}^k g$, $\bigwedge^k(f \circ g) = \bigwedge^k f \circ \bigwedge^k g$.

Proof. Exercise.

Example. Let K be a field and E a finite-dimensional K-vector space of dimension $d \geq 0$. Let $f : E \longrightarrow E$ be a linear map. Then $\overset{d}{\Lambda} f : \overset{d}{\Lambda} E \longrightarrow \overset{d}{\Lambda} E$ is the multiplication by $\det(f)$. In other words, if $(e_1, \dots, e_d)$ is a basis of E, then we have

$$\left(\overset{d}{\Lambda} f\right)(e_1 \wedge \dots \wedge e_d) = (\det f)\,(e_1 \wedge \dots \wedge e_d).$$

The fact that $\det(f \circ g) = \det(f)\det(g)$ becomes simply $\overset{d}{\Lambda}(f \circ g) = \overset{d}{\Lambda}(f) \circ \overset{d}{\Lambda}(g)$.

[To see this, use the characterization of det as the unique alternating d-linear map on E^d such that $\det(\mathrm{Id}) = 1$].

10. Tensor, symmetric and alternating algebras

Let M be an R-module. We can construct

Three R-algebras based on M, which have many applications [but, exceptionally, only one of them is commutative!]

Define

$$T(M) = \bigoplus_{k \geq 0} \underbrace{(M \otimes_R \cdots \otimes_R M)}_{k \text{ times}} \quad : \quad \text{tensor algebra}$$

$$\mathrm{Sym}(M) = \bigoplus_{k \geq 0} \mathrm{Sym}^k(M) \quad : \quad \text{symmetric algebra}$$

$$\Lambda M = \bigoplus_{k \geq 0} \Lambda^k M \quad : \quad \text{exterior algebra}$$

<u>Proposition</u>. $T(M)$, $\mathrm{Sym}(M)$ and ΛM are R-algebras with $\mathrm{Sym}(M)$ commutative, the others usually <u>not</u> with ring structure characterized by

$$\underbrace{(x_1 \otimes - \otimes x_k)}_{\in M^{\otimes k}} \cdot \underbrace{(y_1 \otimes \cdots \otimes y_\ell)}_{\in M^{\otimes \ell}} = \underbrace{x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_\ell}_{\in M^{\otimes (k+\ell)}}.$$

Example. (1) Let M be free of rank $d \geq 0$ with basis $(e_1, \dots, e_d)$. Then we have an iso-morphism $\mathrm{Sym}(M) \xrightarrow{\sim} R[X_1, \dots, X_d]$ of R-algebras, characterized by $e_i \longmapsto X_i$ [here $e_i \in M = \mathrm{Sym}^1(M) \subset \mathrm{Sym}(M)$].

(2) Let M be as above. In ΛM, the product of x and y is denoted $x \wedge y$. We have the rules

$$\begin{cases} e_i \wedge e_j = -e_j \wedge e_i \\ e_i \wedge e_i = 0. \end{cases}$$

So for instance if $d = 4$, then

$$(e_1 \wedge e_3 + e_1 \wedge 2e_2 \wedge 3e_3) \wedge (e_1 - 5\, e_2 \wedge e_4)$$
$$= -5\, e_1 \wedge e_3 \wedge e_2 \wedge e_4 = 5\, e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$