Chapter VI

Dimension : introduction

Goal: define an invariant of a ring, which measures in some way its "complexity" and geometrically corresponds to the "dimension" of the zero set of polynomials (f_i) over a field K when $R = K[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$.

1 - Definition

ACL 8.3

Def. [Krull dimension]

Let R be a ring.

(1) For $k \geq 0$ integer, a prime chain of length k in R is a family

$$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_k$$

of prime ideals in R; we say that this chains is from p_0 to p_k.

(2) Let $p \subset R$ be a prime ideal. The height of p is either the integer $h \in \mathbb{N}$ such that there exists a prime chain in R of length h ending at p and no such chain of length $h+1$; or it is $h = +\infty$. We denote $ht(p)$ the height of p.

(3) The Krull dimension of R is the supremum, in $\mathbb{N} \cup \{+\infty\}$ of the lengths of all prime chains in R. It is denoted $\dim(R)$.

(For $R \neq \{0\}$, $\dim(R) = \sup\limits_{\substack{p \subset R \\ \text{prime}}} ht(p)$.)

2. First examples

Prop. R field $\iff$ (R integral domain and $\dim R = 0$)

Proof. If R is a field then $\{0\} \subset R$ is the only prime ideal, so it has height 0, so $\dim R = 0$.

Conversely, if R is an integral domain of dimension 0, then $\{0\} \subset R$ is prime and is maximal [else $\{0\} \subsetneq M$ would give a chain of length 1], so R is local with $R^\times = R - \{0\}$, i.e., R is a field.

$\square$

<u>Remark</u>. Let $R = K_1 \times K_2$ with K_1, K_2 fields. This is an example of zero-dimensional ring which is not a field.

Later we will classify <u>noetherian</u> zero-dim. rings (these are the "artinian rings").

<u>Prop.</u> If R is a PID, not a field, then $\dim R = 1$, but not conversely, even for integral domains.

<u>Proof</u>. Since R is not a field, there is a prime ideal $p \neq \{0\}$, and

$$\{0\} \subsetneq p$$

[<u>Or</u> use the previous proposition].

shows that $\dim R \geq 1$.

Suppose now that

$$\{0\} \subsetneq \mathfrak{p}_1 \subset \mathfrak{p}_2$$

are prime ideals. Since R is a PID, we can find irreducible elements x_1, x_2 s.t. $\mathfrak{p}_i = x_i R$ for $i = 1, 2$. Then

$$x_1 R \subset x_2 R$$

implies that $x_2 \mid x_1$, so $x_2 = u x_1$ for some unit $u \in R^\times$ since x_1 is irreducible and x_2 is not a unit. So $\mathfrak{p}_2 = \mathfrak{p}_1$, and this shows that there is no longer chain than those of the form $\{0\} \subset \mathfrak{p}$.

$\square$

In particular, examples 1 and 2 show that for a field K, we have

$$\dim K = 0$$

$$\dim K[X] = 1$$

The following result is then maybe not surprising:

Theorem. Let K be a field and $n \in \mathbb{N}$. Then $K[X_1, \_, X_n]$ has finite dimension equal to n. In fact, more generally, if R is a noetherian ring, then

$$\dim R[X_1, \_\_, X_n] = n + \dim R$$

It is easy to check the lower bound

$$\dim R[X_1, \dots, X_n] \geq n + \dim(R)$$

but the converse is more involved, and we will only prove this for fields in a later chapter.

let $A = R[X_1, \dots, X_n]$

Proof (of $\geq$): let $p_0 \underset{\neq}{\subset} \dots \underset{\neq}{\subset} p_m$ be any prime chain in R; then we get

$$p_0 A \underset{\neq}{\subset} \dots \underset{\neq}{\subset} p_m A \underset{\neq}{\subset} p_m A + X_1 A \underset{\neq}{\subset} \dots \underset{\neq}{\subset} p_m A + (X_1, \dots, X_n) A$$

which is a prime chain in A of length $m+n$. [The various terms are prime ideals, e.g. we have

$$R[X_1, \dots, X_n]/_{p_i} \xrightarrow{\sim} \underbrace{(R/_{p_i})[X_1, \dots, X_n]}_{\text{integral domain}}$$

and

$$R[X_1, \dots, X_n]/_{p_m + (X_1, \dots, X_j)} \xrightarrow{\sim} \underbrace{(R/_{p_m})[X_{j+1}, \dots, X_n]}_{\text{integral domain}}$$

for $1 \leq j \leq n$.]

$\square$

<u>Remark.</u> One must restrict to <u>prime chains</u> because otherwise we would get a useless invariant, almost always equal to $+\infty$. [For instance $2^m\mathbb{Z} \subset 2^{m-1}\mathbb{Z} \subset \dots \subset 2\mathbb{Z}$.]

<u>Proposition.</u> Let $p \subset R$ be prime.

(1) We have $ht(p) = \dim(R_p)$

(2) We have

$$\dim(R_p) + \dim(R/p) \leq \dim R.$$

Proof. (1) Let $s: R \longrightarrow R_p$ be the localisation map. We know that there is an order-preserving bijection

$$\begin{cases} \{ q \subset A_p \text{ prime} \} \longrightarrow \{ \tilde{q} \subset p \text{ prime} \} \\ q \longmapsto s^{-1}(q) \end{cases}$$

which formally implies that prime chains in R_p correspond to prime chains in R which end in an ideal in p. Any such chain can be, if needed, lengthened to include the maximal ideal pR_p / the prime ideal p, and the conclusion $\dim(R_p) = ht(p)$ follows.

(2) Let $s: R \longrightarrow R_p$ and $\pi: R \longrightarrow R/p$ be the usual morphisms. Then for any prime chains $p_0 \subsetneq \cdots \subsetneq p_m = pR_p \subset R_p$ and $\{0\} = q_0 \subsetneq \cdots \subsetneq q_k \subset R/p$

we have the prime chain

$$s^{-1}(\mathfrak{p}_0) \subsetneq \cdots \subsetneq s^{-1}(\mathfrak{p}_m) = \mathfrak{p}$$
$$= \pi^{-1}(\mathfrak{q}_0) \subsetneq \pi^{-1}(\mathfrak{q}_1) \subsetneq \cdots \subsetneq \pi^{-1}(\mathfrak{q}_k),$$

which has length $m+k$. [We can assume that $\mathfrak{p}_m = \mathfrak{p}R_\mathfrak{p}$, $\mathfrak{q}_0 = \{0\}$, because these are unique maximal / minimal prime ideals, so any prime chain can be extended to include them to compute dimensions].

$\square$

Note. One does not always have equality in (2), but it happens for most rings; in particular, we will see in Chapter IX that if R is a finitely-generated algebra over a field which is an integral domain.

Example. Let R be a UFD. Let $\{0\} \neq \mathfrak{p} \subset R$ be prime. Then $ht(\mathfrak{p}) = 1 \iff \mathfrak{p}$ is principal:

$\Rightarrow$: There is some irreducible $a \in p$

[indeed, choose $a \in p - \{0\}$ expressible as a product of the fewest possible number of irreducible elements; then we cannot factor a non-trivially, as one factor would be in p and involve fewer irreducibles, so a is irreducible], and $\{0\} \subsetneq aR \subset p$, so $aR = p$ since $ht(p) = 1$.

$\Leftarrow$: let $p = aR$ be prime, so a is an irreducible element; we get $\{0\} \subsetneq aR$. If there is a prime ideal p with

$$\{0\} \subsetneq p \subset aR$$

We must have $a \in p$. Indeed, otherwise, let $b \neq 0$ be an element of p; we can write

$$b = a^d b'$$

where b' is not divisible by a; since p is prime we get $b' \in p$, but $b' \notin aR$, a contradiction. Now $a \in p$ implies $aR \subset p$ so $p = aR$. Therefore $ht(p) = 1$.

$\square$

Remark. We will also prove later two important related results:

(1) [Krull's Principal Ideal Theorem] If R is noetherian and $a \in R$ is not a zero-divisor, then there are minimal prime ideals $p \subset R$ such that $a \in p$, and all these satisfy $ht(p) = 1$.

(2) A noetherian integral domain R is a UFD if and only if every prime ideal $p \subset R$ of height 1 is principal.

Note. There exist examples of infinite-dim. noetherian rings (Nagata). On the other hand, one can show that if R is noetherian and local, then it has finite dimension.