Chapter IX

Finitely-generated algebras over a field

Goal: establish the fundamental properties (e.g the Nullstellensatz and height or dimension results) of what are the most important rings and algebras, at least classically, namely finitely-generated algebras over fields: K-algebras of the form

$$K[X_1, \dots, X_n]/I$$

where $I \subset K[X_1, \dots, X_n]$ is an ideal (necessarily finitely-generated also, since $K[X_1, \dots, X_n]$ is noetherian.

1. Transcendence degree [ACL 4.8]

We begin with some basic material about field extensions which are not algebraic. This will be done without proofs (as this would take a fair amount of time, and is not particularly fun or relevant to the remainder of the lectures).

Def. / properties - Let L/K be a field extension, and $(x_i)_{i\in I}$ a family of elements of L. The following properties are equivalent, and if they are satisfied, one says that the (x_i) are algebraically independent:

(i) The morphism $\begin{cases} K[(X_i)] \longrightarrow L \\ \quad f \longmapsto f((x_i)_{i\in I}) \end{cases}$ is injective.

(ii) There exists a morphism $K((X_i)_{i\in I}) \xrightarrow{\varphi} L$ s.t. $\varphi(X_i) = x_i$ for all $i \in I$. (Here $K((X_i)_{i\in I}) = \mathrm{Frac}(K[(X_i)_{i\in I}]) = \{ f/g \mid f, g \text{ in } K[(X_i)] \})$

Note - If $(x_i)_{i\in I}$ is algebraically independent and $\varphi : K((X_i)) \longrightarrow L$ is the morphism of (ii), its image is denoted $K((x_i)_{i\in I}) \subset L$.

Def. Let L/K be a field extension. A family $(x_i)_{i\in I}$ is a <u>transcendence basis</u> of L over K if (x_i) is algebraically independent and the field extension $L/K((x_i)_{i\in I})$ is algebraic.

Theorem. [ACL 4.8.4, 4.8.5]

Let L/K be a field extension.

(1) There exist transcendence bases $(x_i)_{i\in I}$ of L over K, and the cardinality of the set I is independent of the transcendence of basis.

(2) More precisely, if $A \subset C$ are subsets of L with A algebraically independent and C such that L is algebraic over $K(C)$, then there exists a transcendence basis B such that $A \subset B \subset C$.

Note. There is a clear analogy here with linear algebra :

Field theory	Linear algebra
L/K	V K-vector space
algebraic independance	linear independance
$L / K((x_i))$ algebraic	generating family
transcendance basis	basis of V

So the second part of the theorem is just analogue of "given $A \subset C$ with A linearly independent and C generating set of V, we can find a basis B such that $A \subset B \subset C$".

In fact, although it is not the simplest approach, one can actually develop a more general theory (based on "model theory") so that both settings are included and common proofs are found.

Def. The transcendence degree of L/K, denoted

$$\mathrm{trdeg}\,(L/K) = \mathrm{trdeg}_K(L),$$

is the cardinality of any transcendance basis of L/K.

Example - (1) $\mathrm{Trdeg}(L/K) = 0 \iff L$ is algebraic over K.

(2) If $(x_i)_{i \in I}$ are indeterminates, then

$$\mathrm{trdeg}\left(K((x_i)_{i \in I})/K\right) = \mathrm{Card}(I)$$

[also for I infinite].

(3) If $x \in L$ then $\mathrm{trdeg}(K(x)/K) \leq 1$, with equality if and only if x is transcendental over K.

In particular, if $(x_i)_{i \in I}$ are algebraically independent, then each x_i is transcendental over K. The converse is false: for instance (π, π^2) are elements of $\mathbb{R}$, each of which is transcendental, but they are not algebraically independent [since $f(\pi, \pi^2) = 0$ where $f = X_2 - X_1^2$].

(4) Let $L = \mathrm{Frac}\left(\mathbb{C}[X,Y]/(Y^2 - X^3 - X)\right)$. Then $\mathrm{trdeg}(L/\mathbb{C}) = 1$. Indeed, first note

that $Y^2 - X^3 - X \in \mathbb{C}[X,Y]$ is irreducible, so the field L makes sense. Moreover, one can check that X is algebraically independent (in other words $\mathbb{C}[X] \longrightarrow \mathbb{C}[X,Y]/(Y^2 - X^3 - X)$ is injective), and then L is an <u>algebraic extension</u> of $\mathbb{C}(X) \subset L$ [because in fact

$$L = \mathbb{C}(X)[Y] / (Y^2 - (X^3 + X))$$

so Y is algebraic over $\mathbb{C}(X)$]. Hence we get $\mathrm{trdeg}(L/\mathbb{C}) = 1$.

(5) If L_1 and L_2 are algebraically closed fields of characteristic 0 with $\mathrm{trdeg}(L_1/\mathbb{Q}) = \mathrm{trdeg}(L_2/\mathbb{Q})$ then L_1 and L_2 are isomorphic: indeed L_1 and L_2 are both isomorphic to an algebraic closure of the field $\mathbb{Q}((x_i)_{i\in I})$, where $\mathrm{Card}(I) = \mathrm{trdeg}(L_i/\mathbb{Q})$.

For instance, the fields $L_1 = \mathbb{C}_2$ and $L_2 = \overline{\mathbb{Q}_\ell}$ [algebraic closure of the field of ℓ-adic numbers, which will be defined in Ch. X] are isomorphic, because one can prove that they have same tr. degree.

As a last bit of terminology: a field extension of the form $K((X_i)_{i\in I})/K$, with $(X_i)_{i\in I}$ indeterminates, is called a purely transcendental extension. (Note that its transcendence degree is the cardinality of I, but the converse is not true).

2 - Noether's Normalization Theorem

We fix a base field K.

Theorem - [ACL 9.1.1]

Let R be a non-zero finitely-generated K-algebra. There is an integer $n \geq 0$ and a commutative diagram:

$$\begin{tikzcd} t \arrow[d, mapsto] & K \arrow[r] \arrow[d] & R \\ t & K[X_1,\dots,X_n] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

structure morphism, injective because $R \neq \{0\}$

where φ is injective and integral.

<u>Note</u>. We will see later that if R is an integral domain, then $n = \operatorname{trdeg}(\operatorname{Frac}(R)/K) = \dim(R)$.

<u>Proof</u>. We argue by induction on $m \geq 0$ such that $R = K[a_1, \dots, a_m]$ for some (a_i) in R^m. If $m = 0$, then $R = K$ so we can take $n = 0$, $\varphi = \mathrm{Id}_K$.

Suppose now that the statement holds for all algebras over K generated by $\leq m-1$ elements.

Consider the morphism $K[X_1, \dots, X_m] \xrightarrow{\alpha} R$ such that $\alpha(X_i) = a_i$; it is surjective. If it is injective then we are done, taking $n = m$ and $\varphi = \alpha$.

So suppose that there exists $f \in K[X_1, \dots, X_m]$ such that $f(a_1, \dots, a_m) = 0$. We view f as a poly-nomial $f \in K[X_2, \dots, X_m][X_1]$, and write

$$f = \sum_{i=0}^{d} f_i \, X_1^i$$

where $f_i \in K[X_2, \dots, X_m]$.

Observe then that if f <u>is monic</u> $[\text{i.e. } f_d = 1]$

Then the equation $f(a_i) = 0$ means that $a_1 \in R$ is integral over $A = K[a_2, \dots, a_m]$. Since A is a K-algebra generated by $m-1$ elements we can find by induction a commutative diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & A \\ K[X_1, \dots, X_n] \arrow[ur, "\varphi_1"'] & \end{tikzcd}$$

with φ_1 injective and integral. Then we get a diagram

$$\begin{tikzcd} K \arrow[r] \arrow[dr] & A \arrow[r, "i"] & R \\ & K[X_1, \dots, X_n] \arrow[u, "\varphi_1"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

(inclusion)

, with $\varphi = i \circ \varphi_1$, and $K[X_1, \dots, X_n] \xrightarrow{\varphi} R$ injective and integral and we are done.

<u>However</u>, there is a priori no reason for the equation f to be monic with respect to X_1, or with respect to any variable.

In order to make this argument work, we will then exploit the fact that polynomial rings have many automorphisms, and that for any auto-

-morphism σ of $K[X_1, \dots, X_n]$, we can use (b_i) instead of (a_i), where $b_i = \sigma(a_i)$; we will see that for <u>some</u> σ, the tuple (b_i) has the desired property that b_1 is <u>integral</u> over $K[b_2, \dots, b_m]$. Finding σ is easier if <u>K is infinite</u>, so we assume this (giving the answer for K finite at the end, without proving that it works).

Let
$$\begin{cases} \sigma(X_1) = X_1 \\ \sigma(X_i) = X_i - c_i X_1, \quad i \geq 2 \end{cases}$$

where $c_2, \dots, c_m$ are coefficients in $K^\times$ to be chosen later. Note that
$$X_i = c_i X_1 + \sigma(X_i)$$
$$= \sigma(X_i + c_i X_1)$$

so σ is indeed an automorphism. Defining as above $b_i = \sigma(a_i)$, an equation satisfied by (b_i) is $\tilde{f}(b_i) = 0$ where
$$\tilde{f} = f(X_1, (c_2 X_1 + X_2), \dots, (c_m X_1 + X_m)).$$

Using the previous expression $f = \sum_{i=0}^{d} f_i X_1^i$,

we get

$$\tilde{f} = \sum_{i=0}^{d} \underbrace{f_i\Big((c_2 X_1 + X_2), \dots\Big)}_{\tilde{f}_i} X_1^i .$$

Let $M = \alpha X_2^{d_2} \dots X_m^{d_m} X_1^i$ be a monomial appearing in f_1. In $\tilde{f}$, it becomes

$$\alpha (c_2 X_1 + X_2)^{d_2} \dots (c_m X_1 + X_m)^{d_m} X_1^i$$
$$= \alpha\, c_2^{d_2} \dots c_m^{d_m} X_1^{d_2 + \dots + d_m + i} + (\text{lower degree in } X_1)$$

Summing over the monomials appearing in f, we deduce that the polynomial $\tilde{f}$ has degree in X_1 equal to the total degree δ of f, and that the term of degree δ in X_1 is equal to

$$\underbrace{\left(\sum_{\substack{\text{monomials} \\ \text{in } f \text{ of degree } \delta}} \alpha\, 1^i c_2^{d_2} \dots c_m^{d_m} \right)}_{\beta} X_1^{\delta}$$

We are done if $\beta \neq 0$ (since the leading term is then a non-zero constant). But we see that

$\beta = g(1, c_2, \dots, c_m)$ where g is the homogeneous part of f of degree δ. We have $g \neq 0$ (since $f \neq 0$) and because K is infinite, it is standard that some value of g is non-zero; and since g is homogeneous, some value $g(1, c_2, \dots, c_m)$ is non-zero. So we are done for K infinite. In general, we use a different σ, namely

$$\begin{cases} \sigma(X_1) = X_1 \\ \sigma(X_i) = X_i - X_1^{r^{i-1}}, & i \geqslant 2 \end{cases}$$

where now the only parameter $r \geqslant 2$ is a (large enough) integer.

Here a monomial $X_2^{d_2} \dots X_m^{d_m}$ is transformed into

$$X_1^{\sum_{i=2}^{m} d_i r^{i-1}}$$

and the point is that by uniqueness of base-r expansion, the exponents will be different for each monomial if r is larger than all the d_i's that occur in the polynomial f. From this, it is not hard to deduce that b_1 is integral over $K[b_2, \ldots, b_m]$, as before.

$\square$

<u>Note</u> – There is no uniqueness in the decomposition given by Noether's Normalization. This can be a problem in some applications.

3 – Zariski's Theorem and Hilbert's Nullstellensatz

Noether's Theorem leads to short proofs of various forms of the Nullstellensatz.

First comes a result of Zariski:

<u>Theorem</u> (Zariski) ACL 9.1.2

Let K be a field and R a finitely-generated K-algebra. If R is a field, then it is an <u>algebraic</u> extension of K; it is in fact a finite extension.

<u>Proof.</u> We have $R \neq \{0\}$ if R is a field so we get the Noether decomposition

$$\begin{tikzcd} K \arrow[rr] \arrow[dr] & & R \\ & K[x_1,\dots,x_n] \arrow[ur, "\varphi"'] & \end{tikzcd}, \quad \varphi \text{ injective and integral.}$$

By "going-up", we know that

$$0 = \dim(R) = \dim(K[x_1,\dots,x_n])$$

which is only possible if $n = 0$ (for instance because $\dim K[x_1,\dots,x_n] \geq n$), so that R is an integral extension of K and a field, hence an algebraic extension. Since it is finitely-generated, it is a finite extension.

$\square$

<u>Corollary</u> ("weak" Nullstellensatz) ACL 9.1.3

Let K be an <u>algebraically closed</u> field and $n \geq 1$. The maximal ideals of $K[x_1,\dots,x_n]$ are the ideals of the form $(x_1 - a_1, \dots, x_n - a_n)$ for $(a_i) \in K^n$.

Proof. The isomorphism

$$\begin{cases} K[x_1, \dots, x_n] \big/ (x_1 - a_1, \dots, x_n - a_n) \longrightarrow K \\ f \longmapsto f(a_1, \dots, a_n) \end{cases}$$

implies that these ideals are all maximal. [it is injective because if $f(a_i) = 0$, the Taylor expansion around (a_i) has the form

$$f = f(a_i) + \sum_i \partial_{x_i} f(a_i)\,(x_i - a_i) + \dots$$

$$\in (x_1 - a_1, \dots, x_n - a_n)\ ]$$

Conversely, let $m \subset K[x_1, \dots, x_n]$ be a maximal ideal. Then $K[x_1, \dots, x_n]/m$ is a finitely-generated K-algebra which is a field, so it is a (finite) algebraic extension of K, which can only be K (since K is alg. closed) by Zariski's Theorem. But then let $a_i \in K$ be such that

$$x_i - a_i \in m$$

(i.e. $a_i \in K$ projects mod m to the class of X_i mod m) ; we find that $(X_1 - a_1, \dots, X_n - a_n)$ is contained in m, and because of the first part, this ideal is maximal, so is equal to m.

Finally, we observe that the ideals associated to $(a_i) \neq (b_i)$ are <u>different</u> : if $a_i \neq b_i$, then

$X_i - a_i \in (X_1 - a_1, \dots, X_n - a_n)$ but

$X_i - b_i \equiv a_i - b_i \neq 0 \mod (X_1 - b_1, \dots, X_n - b_n)$.

$\square$

<u>Cor.</u> Let K be alg. closed, $I \subset K[X_1, \dots, X_n] = R$ an ideal. Let $X = \{ (a_i) \in K^n \mid f((a_i)) = 0 \text{ for all } f \in I \} \subset K^n$.

Then $X = \emptyset \iff I = R$.

[which means, if I is generated by $f_1, \dots, f_m$, that we can write $1 = \sum_i f_i g_i$ for some polynomials (g_i)]

Proof. If $I = R$ then $1 \in I$ so $X = \emptyset$.
On the other hand, suppose $I \neq R$; let then m be a maximal ideal such that $I \subset m$; let $(a_1, \dots, a_n) \in K^n$ be such that $m = (X_1 - a_1, \dots, X_n - a_n)$; since the projection $K[X_1, \dots, X_n] \xrightarrow{\pi} K[X_1, \dots, X_n]/m$ is given by $\pi(f) = f(a_1, \dots, a_n)$, and since $\pi(I) = \{0\}$, we deduce that $(a_i) \in X$.

$\square$

Note. The proof in fact shows that X is in bijection with the set of maximal ideals containing I. This is the beginning of "abstract" algebraic geometry.

The following facts are also important:

Proposition. Let K be a field, and let R and A be finitely-generated K-algebras.

(1) For any morphism $R \xrightarrow{f} A$ of K-algebras and $m \subset A$ maximal ideal, the ideal $f^{-1}(m)$ is maximal in R.

(2) The Jacobson radical of R is equal to its nilpotent radical; more generally, for any ideal $I \subset R$, we have

$$\sqrt{I} = \bigcap_{\substack{m \supset I \\ m \text{ max.}}} m$$

where $\sqrt{I} = \{ f \in R \mid \exists\, m \geq 0,\ f^m \in I \}$.

Proof. (1) We have a morphism

$$K \longrightarrow R/f^{-1}(m) \longrightarrow A/m$$

; by Zariski's Th., A/m is a finite extension of K; but then $R/f^{-1}(m)$ is a finite-dimensional K-vector space, which implies that it is a field (because for $x \neq 0$, $y \longmapsto xy$ is an injective K-linear map $R/f^{-1}(m) \longrightarrow R/f^{-1}(m)$, so it is surjective, hence 1 is in the image). So $f^{-1}(m)$ is maximal.

(2) The nilpotent radical $\bigcap_{p \text{ prime}} p$ is contained in the Jacobson radical $\bigcap_{m \text{ max}} m$. Conversely, suppose $a \notin \bigcap_{p \text{ prime}} p$. Then a is not nilpotent, which means that the morphism $R \xrightarrow{f} R_a$ is injective. Note that $R_a \simeq R[X]/(1-aX)$ is again finitely-generated as a K-algebra (but this wouldn't be true for many other localizations of R!). Let $m \subset R_a$ be a maximal ideal; by (1), the ideal $m' = f^{-1}(m)$ in R is maximal, and it does not intersect the multiplicative set $\{1, a, a^2, \dots\}$, in particular $a \notin m'$, so a is not in the Jacobson radical.

The case of $\sqrt{I}$ follows because $\sqrt{I}$ is the inverse image $\pi^{-1}(\text{nilp. radical})$ for $\pi : R \to R/I$ the projection, and $\bigcap_{m \supset I} m$ is $\pi^{-1}\left(\begin{array}{c}\text{Jacobson}\\ \text{radical}\end{array}\right)$, and R/I is again a finitely-generated K-algebra.

$\square$

<u>Corollary</u> ("strong" Nullstellensatz)

Assume K is algebraically closed. Let $n \geq 1$ and $I \subset K[X_1, \dots, X_n]$ an ideal. Let

$$V = \{ (x_i) \in K^n \mid f(x) = 0 \text{ for all } f \in I \}.$$

If $g \in K[X_1, \dots, X_n]$ satisfies $g(x) = 0$ for all $x \in V$, then some g^m, $m \geq 0$, is in I.

(i.e., $g \in \sqrt{I}$)

<u>Proof</u>. In view of the Nullstellensatz, the condition on g is that $g \in \bigcap_{\substack{m \text{ max} \\ m \supset I}} m = \sqrt{I}$ by the proposition.

$\square$

<u>Remark</u>. If $V = \emptyset$, this says $1 \in \sqrt{I}$ so $1 \in I$, recovering the corollary of the "weak" version.

4 _ Height and dimension

We are now able to prove the key properties of height and dimension for f.g. K-algebras.

<u>Theorem</u> - (1) For all $n \geq 0$, we have

$$\dim K[x_1, \dots, x_n] = n.$$

(2) If R is a f.g. K-algebra and is an integral domain, then (ACL 9.3.11)

$$\dim R = \operatorname{trdeg}_K(\operatorname{Frac}(R))$$

(3) If R is a f.g. K-algebra and is an integral domain then $ht(p) + \dim(R/p) = \dim(R)$ for any prime ideal $p \subset R$.

<u>Proof</u>. Note that (2) implies (1) by taking $R = K[x_1, \dots, x_n]$. So we prove (2) by induction on $t = \operatorname{trdeg}_K(\operatorname{Frac}(R))$. If $t = 0$, then $\operatorname{Frac}(R)$ is a finite (algebraic) extension of R, so R is a field and has dimension 0.

Now assume that $t \geq 1$ and the result holds for smaller tr. degree. By Noether's Theorem, we get

$$\begin{tikzcd} K \arrow[r] \arrow[dr] & R \\ & K[x_1,\dots,x_n] \arrow[u, "\varphi"'] \end{tikzcd}$$

with φ integral, so

$$\dim R = \dim K[x_1,\dots,x_n] \geq n.$$

$$\operatorname{tr\,deg}_K(\operatorname{Frac}(R)) = \operatorname{trdeg}_K(K(x_1,\dots,x_n)) = n.$$

Let $\{0\} = p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_m$ be a chain in $K[x_1,\dots,x_n]$. Let

$$A = K[x_1,\dots,x_n]/p_1.$$

Then we have a chain of length $m-1$ in A by reducing mod p_1.

On the other hand, take $f \neq 0$ in p_1; then $f(x_1,\dots,x_n) = 0$ in A, i.e., the classes of $x_1,\dots,x_n$ are algebraically dependent in $L = \operatorname{Frac}(A)$. Since these classes generate L over K, we must have $\operatorname{trdeg}_K(L) \leq n-1$.

Hence by induction we have $\dim(A) \leq n-1$ so $m-1 \leq n-1$; from $m \leq n$, we deduce that $\dim R = \dim K[x_1, \dots, x_n] \leq n$, hence the result.

(3) We proceed by induction on $\dim(R)$.
If $\dim(R) = 0$, then R is a field, so $p = (0)$ is the only prime ideal, and $ht(p) = \dim(R/p) = 0$.
Suppose now that $\dim(R) \geq 1$ and the statement holds for K-algebras of dimension $< \dim(R)$.
We apply Noether's Theorem to get

$$\begin{tikzcd} K \arrow[rr] \arrow[dr] & & R \\ & K[x_1, \dots, x_n] \arrow[ur, "\varphi"] & \end{tikzcd}$$

with φ integral

(so $n = \dim(K[x_1, \dots, x_n]) = \dim(R)$).

By the "going-down" theorem, we also have $ht_R(p) = ht_A(\varphi^{-1}(p))$, where $A = K[x_1, \dots, x_n]$.
Moreover, since $A/\varphi^{-1}(p) \hookrightarrow R/p$ is also

integral, we have $\dim(R/p) = \dim(A/\varphi^{-1}(p))$, and altogether, this means that we can reduce to the case $R = A = K[x_1, \dots, x_n]$.

Case 1 - If $p \subset A$ is zero, then $ht(p) = 0$ and the equality holds

Case 2 - If $p \neq \{0\}$, then we find $f \in p$ irreducible (some $a \neq 0$, $a \notin A^\times$ is in p, so some irreducible factor of a is in p). The principal ideal fA has height 1 (because A is a UFD).

Define $B = A/_{fA}$, $p' = p/_{fA} \subset B$, so that p' is a prime ideal in B.

By the lemma below we have also

$$\mathrm{trdeg}_K B = \mathrm{trdeg}_K A - 1 = n-1 ;$$

moreover $B/_{p'} = A/_{fA} \big/ _{p/_{fA}} \simeq A/_p$ so

$$\dim(B/_{p'}) = \dim(A/_p) \quad ;$$

and finally

$$\mathrm{ht}_B(\mathfrak{p}') \leq \mathrm{ht}_A(\mathfrak{p}) - 1.$$

Using the induction hypothesis, we know that

$$\mathrm{ht}_B(\mathfrak{p}') + \dim(B/\mathfrak{p}') = \dim(B)$$

so

$$\begin{aligned}
\mathrm{ht}_A(\mathfrak{p}) + \dim(A/\mathfrak{p}) &\geq 1 + \mathrm{ht}_B(\mathfrak{p}') + \dim(B/\mathfrak{p}') \\
&= 1 + \dim(B) \\
&= 1 + \mathrm{trdeg}_K(B) \\
&= 1 + n - 1 = \dim(A),
\end{aligned}$$

concluding the proof.

$\square$

We used the following lemma:

<u>Lemma</u>. Let $n \geq 1$ and f irreducible in $K[X_1, \dots, X_n]$. Then $\mathrm{trdeg}_K\left(\mathrm{Frac}\left(K[X_1, \dots, X_n]/(f)\right)\right) = n - 1$.

<u>Proof</u>. Since f is irreducible, it is not constant, so (by permuting the variables if necessary) we can assume that X_n occurs in f with degree ≥ 1.

We then claim that the classes $Y_1, \dots, Y_{n-1}$ of the variables $X_1, \dots, X_{n-1}$ in $K[X_1, \dots, X_n]/(f)$ form a transcendence basis of the fraction field L, this proves the lemma.

Indeed, the equation $f(Y_1, \dots, Y_n) = 0$ shows that the $(Y_i)_{1 \leq i \leq n}$ are algebraically dependent; since $Y_1, \dots, Y_n$ generate $K[X_1, \dots, X_n]/(f)$ as a K-algebra, this implies that L is algebraic over $K(Y_1, \dots, Y_{n-1})$. Moreover, we claim that the morphism

$$\left\{ \begin{array}{ccc} K[X_1, \dots, X_{n-1}] & \longrightarrow & L \\ X_i & \longmapsto & Y_i \end{array} \right.$$

is injective: if g is in the kernel then $g(Y_1, \dots, Y_{n-1}) = 0$ means that $g \in K[X_1, \dots, X_{n-1}]$ is a multiple of f. But since $\deg_{X_n} f \geqslant 1$, this is only possible if $g = 0$.

So $(Y_1, \dots, Y_{n-1})$ are algebraically independent over K; it is therefore a transcendence basis. $\square$