D-MATH
HS 2021
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Exercise sheet 1

Commutative Algebra

(1) Let $\mathscr{C}$ be the category of vector spaces over $\mathbb{C}$ and $\mathscr{D}$ the category of sets. Let F be the forgetful functor $\mathscr{C} \to \mathscr{D}$.

a. Show that one can define a functor G from $\mathscr{D}$ to $\mathscr{C}$ by $G(X) = \mathbb{C}^{(X)}$ (the vector space with basis X, or the vector space of functions from X to $\mathbb{C}$ which are zero for all but finitely many $x \in X$), with $G(f\colon X \to Y)$ the linear map $\mathbb{C}^{(X)}$ to $\mathbb{C}^{(Y)}$ such that the basis vector $x \in X$ of $\mathbb{C}^{(X)}$ is mapped to the basis vector $f(x) \in Y$.

b. Show that for any set X and vector space V, there is a bijective map of sets

$$\mathrm{Hom}_{\mathscr{C}}(G(X), V) \to \mathrm{Hom}_{\mathscr{D}}(X, F(V)).$$

(2) A functor $F\colon \mathscr{C} \to \mathscr{D}$ is said to be *fully faithful* if for any objects X and Y of $\mathscr{C}$, the map of sets

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$$

defined by $f \mapsto F(f)$ is a bijection.

Which of the following functors are fully faithful?

a. The forgetful functor from groups to sets.

b. The forgetful functor from topological spaces to sets.

c. The unit functor from rings to groups.

d. The duality functor from $\mathbb{C}$-vector spaces to the opposite category of $\mathbb{C}$-vector spaces.

e. The duality functor from finite-dimensional $\mathbb{C}$-vector spaces to the opposite category of finite-dimensional $\mathbb{C}$-vector spaces.

(3) A functor $F\colon \mathscr{C} \to \mathscr{D}$ is said to be *essentially surjective* if for any object Y of $\mathscr{D}$, there exists an object X of $\mathscr{C}$ such that Y is *isomorphic* to $F(X)$.

Which of the following functors are essentially surjective?

a. The forgetful functor from groups to sets.

b. The forgetful functor from topological spaces to sets.
c. The unit functor from rings to groups.
d. The duality functor from $\mathbb{C}$-vector spaces to the opposite category of $\mathbb{C}$-vector spaces.
Hint: you can use the fact that if V is an infinite-dimensional vector space, then the dual space $V^* = \{f : V \to \mathbb{C} : f\ \mathbb{C}-\text{linear}\}$ has strictly larger dimension (in the sense of infinite cardinals).
e. The duality functor from finite-dimensional $\mathbb{C}$-vector spaces to the opposite category of finite-dimensional $\mathbb{C}$-vector spaces.

(4) Let $\mathscr{C}$ be the category with objects the set $\mathbf{N}$ of natural numbers and with

$$\mathrm{Hom}_{\mathscr{C}}(m, n) = M_{n,m}(\mathbb{C}),$$

the set of matrices with n rows and m columns.

a. Check that with $\mathrm{Id}_n = 1_n$, the identity matrix, and with composition given by the product of matrices, this is indeed a category.
b. Let $(fdVec)$ be the category of finite-dimensional $\mathbb{C}$-vector spaces (with linear maps as morphisms). Show that $F\colon \mathscr{C} \to (fdVec)$ defined by

$$F(n) = \mathbb{C}^n \qquad F(A) = (x \mapsto Ax)$$

(where A is a matrix) is a functor.
c. Show that F is fully faithful and essentially surjective. (Such a functor is called an *equivalence of categories*; it means that, from many points of view, the categories $(fdVec)$ and $\mathscr{C}$ behave in the same way, although the sets of objects and morphisms are very different).
d. Define a functor $G\colon (fdVec) \to \mathscr{C}$ and natural transformations $F \circ G \to \mathrm{Id}_{(fdVec)}$ and $F \circ G \to \mathrm{Id}_{(fdVec)}$ (i.e., natural transformations

$$F(G(V)) \to V, \qquad G(F(n)) \to n$$

for all objects V of $(fdVec)$ and n of $\mathscr{C}$).

Hint: you can first use the axiom of choice to fix a basis of any object V of $(fdVec)$.