D-MATH
HS 2021
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Exercise sheet 2

Commutative Algebra

1 Let p be a prime number and let R be the local ring $\mathbb{Z}_{p\mathbb{Z}}$. Let k be the residue field of R. Construct an isomorphism $\mathbb{Z}/p\mathbb{Z} \to k$.

2 Let R be a ring and $S \subset R$ a multiplicative subset.

(a) If $S \subset R^\times$, show that the canonical morphism

$$\varphi_S \colon R \to S^{-1}R$$

is an isomorphism.

(b) Conversely, if φ_S is an isomorphism, show that $S \subset R^\times$.

3 Let R be a ring and $S \subset R$ a multiplicative subset. Let A be the polynomial ring $A = R[(X_s)_{s\in S}]$ with one variable for each element of S. Let I be the ideal of A generated by all the elements $Y_s = 1 - sX_s \in A$ for $s \in S$. Let $B = A/I$ be the quotient ring. Let $\psi \colon R \to B$ be the canonical morphism (composition of $R \to A \to A/I$) that makes B into an R-algebra.

(a) Show that for any R-algebra T and any R-algebra morphism $f \colon B \to T$, the composition $f \circ \psi \colon R \to T$ satisfies $(f \circ \psi)(S) \subset T^\times$.

(b) Conversely, if $g \colon R \to T$ is a morphism of R-algebras, and $g(S) \subset T^\times$, show that there exists a morphism $f \colon B \to T$ of R-algebras such that $g = f \circ \psi$.

Hint: either construct f "by hand", or use the characteristic properties of morphisms from polynomial rings and from quotient rings.

(c) Show that the map $f \mapsto f \circ \psi$ gives a bijection

$$\mathrm{Hom}_{R-\mathrm{alg}}(B, T) \longrightarrow \{g \colon R \to T \mid g(S) \subset T^\times\}.$$

(d) Deduce that B is isomorphic to $S^{-1}R$, and write an explicit isomorphism $B \to S^{-1}R$.

4 Let K be a field and let $A = K[[X]]$ be the ring of *formal power series with coefficients in* K, i.e., of formal expressions

$$a = \sum_{n \geq 0} a_n X^n$$

with the usual rules for sums and products, e.g.

$$\Big(\sum_{n\geq 0} a_n X^n\Big) \cdot \Big(\sum_{n\geq 0} b_n X^n\Big) = \sum_{n\geq 0} c_n X^n,$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

(a) Show that A is a local ring with maximal ideal

$$m_A = \Big\{\sum_{n=0}^{+\infty} a_n X^n \mid a_0 = 0\Big\}.$$

What is the residue field $k = A/m_A$?

(b) Show that A is principal, in fact that any non-zero ideal I in A is of the form $I = (X^n)$ for some integer $n \geq 0$.

Hint: given a non-zero ideal I, consider an element $a \in I$ with smallest number of coefficients a_0, a_1, ..., equal to 0.

(c) What is the inverse in A of $1 + X$?

5 Let A be a ring and consider the polynomial ring $A[X]$. Let $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a polynomial. Prove that:

(a) f is a unit in $A[X]$ if and only if a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent;

(b) f is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent;

(c) f is a zero-divisor if and only if there exists $a \neq 0$ in A such that $af = 0$.

Furthermore, prove that in the ring $A[X]$, the Jacobson radical is equal to the nilradical.

6 Let A be a ring and let S be a multiplicative subset of A. Prove that S is a multiplicative subset of $A[X]$, and construct an isomorphism of rings

$$S^{-1}(A[X]) \longrightarrow (S^{-1}A)[X].$$