D-MATH
HS 2021
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Exercise sheet 3

Commutative Algebra

1 Let R be a ring and $I \subset R$ the nilradical of R.

a. Show that if R is noetherian, there exists an integer $k \geq 0$ such that $x^k = 0$ for all $x \in I$.
b. Show that this is not true for all rings.

2 Let R be a noetherian ring and $S \subset R$ a multiplicative set. Show that $S^{-1}R$ is noetherian. Give an example (where S contains no zero-divisor) where the converse does not hold.

3 Let R be a ring and M a noetherian R-module. Let

$$I = \{r \in R \mid rM = \{0\}\}.$$

Show that the quotient ring R/I is noetherian.
Hint: construct an injective R-linear map $R/I \to M^n$ for some integer $n \geq 0$.

4 Let R be the local ring $\mathbb{Z}_{p\mathbb{Z}}$.

a. Show that $\mathbb{Q}$ is an R-module.
b. Show that $\mathbb{Q} = J\mathbb{Q}$ where J is the Jacobson radical of R.

5 Let R be the set of $f \in \mathbf{Q}[X]$ such that $f(\mathbf{Z}) \subset \mathbf{Z}$.

a. Check that the set R is a subring of $\mathbf{Q}[X]$ containing $\mathbf{Z}[X]$.
b. Let $f\colon \mathbf{Z} \to \mathbf{Z}$ be a function. If there exists $g \in R$ such that $f(n) = g(n+1) - g(n)$ for all $n \in \mathbf{Z}$, then there is an $f' \in R$ so that $f'(n) = f(n)$ for all $n \in \mathbb{Z}$.
c. Show that the functions f_k defined for $k \geq 0$ by

$$f_k(n) = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

are in R, and that R is a free $\mathbf{Z}$-module with basis $(f_k)_{k\geq 0}$.
d. Show that R is not noetherian.

6 Let R be a ring and M a noetherian R-module. Let $u\colon M \to M$ be an R-linear map. Show that u is surjective if and only if u is an isomorphism.
Hint: consider the submodules $\ker(u^n)$.

7 Let A be a commutative ring which is not noetherian. Let $\mathscr{F}$ be the set of all ideals of A which are not finitely generated, so that $\mathscr{F} \neq \emptyset$.

a. Prove that $\mathscr{F}$ is an inductive set for the inclusion relation (i.e., any totally ordered subset has an upper-bound). Let P be a maximal element of $\mathscr{F}$. In the rest of the exercise, we shall prove that P is a (non-finitely generated) prime ideal of A. We argue by contradiction. Let $a, b \in A$ be such that $ab \in P$ while $a \notin P$ and $b \notin P$.

b. Prove that there exist $u_1, \ldots, u_m \in P$ and $v_1, \ldots, v_n \in A$ such that $P + (a) = (u_1, \ldots, u_m, a)$ and $(P : (a)) = (v_1, \ldots, v_n)$, where

$$(P : (a)) = \{x \in A \mid xa \in P\}.$$

c. Prove that $P = (u_1, \ldots, u_m, av_1, \ldots, av_n)$. Derive from this contradiction that P is a prime ideal of A.

This proves that a commutative ring is noetherian if and only if every prime ideal is finitely generated, a theorem of I. S. Cohen.