D-MATH
HS 2021
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Exercise sheet 4

Commutative Algebra

1 Let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$, and consider the isomorphism of $\mathbb{R}$-vector spaces

$$\begin{aligned} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 &\xrightarrow{\cong} \mathbb{R}^4 \\ e_1 \otimes e_1 &\longmapsto f_1 \\ e_1 \otimes e_2 &\longmapsto f_2 \\ e_2 \otimes e_1 &\longmapsto f_3 \\ e_2 \otimes e_2 &\longmapsto f_4. \end{aligned}$$

a. Show that $af_1 + bf_2 + cf_3 + df_4 \in \mathbb{R}^4$ is the image of a pure tensor $x \otimes y \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ if and only if

$$ad = bc.$$

b. Compute the matrix $u_1 \otimes u_2$, where

$$u_1 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

and

$$u_2 = \begin{pmatrix} -1 & 4 \\ 2 & 3 \end{pmatrix}.$$

2 Show that $2 \otimes 1 = 0$ in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, but $2 \otimes 1 \neq 0$ in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

3 Let A be a commutative ring and B an A-algebra via $s : A \to B$. Let M, M' be B-modules. An A-**derivation**

$$d : B \longrightarrow M$$

of M is a morphism of abelian groups such that

$$\begin{aligned} d(bb') &= bd(b') + b'd(b), \\ d(s(a)) &= 0 \end{aligned}$$

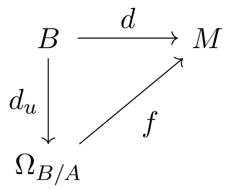
for all $b, b' \in B$, $a \in A$.

a. Prove that for any B-linear map $f : M \to M'$, the composition $f \circ d$ is an A-derivation on M'.

b. Prove that there exists a *universal A-derivation*, i.e. a B-module $\Omega_{B/A}$ with an A-derivation

$$d_u : B \longrightarrow \Omega_{B/A}$$

with the property that for any B-module M and any A-derivation $d : B \to M$ there is a unique B-linear map $f : \Omega_{B/A} \longrightarrow M$ so that the following diagram commutes:



Hint: Construct $\Omega_{B/A}$ as a quotient of a free module.

4 Let M_1, M_2, N_1, N_2 be A-modules.

a. Define an A-linear morphism

$$\mathrm{Hom}_A(M_1, N_1) \otimes_A \mathrm{Hom}_A(M_2, N_2) \xrightarrow{F} \mathrm{Hom}_A(M_1 \otimes_A M_2, N_1 \otimes_A N_2).$$

b. If $A = K$ is a field and M_1, M_2, N_1, N_2 are finite dimensional over K, then F is an isomorphism.

c. Consider $A = \mathbb{Z}/4\mathbb{Z}$, $I = 2\mathbb{Z}/4\mathbb{Z} \subseteq A$ and $M_1 = A/I$, $M_2 = A$, $N_1 = A$, $N_2 = A/I$. Show that in this case F is not injective and not surjective.

d. Compute the $\mathbb{R}$-vector space

$$\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}^m$$

for every $m, n \geq 1$.

5 Let R be a ring and let M', M, M'' and N', N, N'' be R-modules with short exact sequences

$$0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$

$$0 \to N' \xrightarrow{a} N \xrightarrow{b} N'' \to 0$$

denoted (M) and (N) for short.

A *morphism* $(M) \to (N)$ is the data of three linear maps $f' \colon M' \to N'$, $f \colon M \to N$, $f'' \colon M'' \to N''$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{a} & N & \xrightarrow{b} & N'' & \longrightarrow & 0 \end{array}$$

commutes, i.e. $f \circ u = a \circ f'$ and $f'' \circ v = b \circ f$.

a. Define the identity morphism $(M) \to (M)$ so that we obtain a category of short exact sequences.

b. Show that any short exact sequence is isomorphic to an exact sequence of the form

$$0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$$

where i is the inclusion of a submodule and π the canonical projection.

6 Let R be a ring and let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{a} & N & \xrightarrow{b} & N'' & \longrightarrow & 0 \end{array}$$

be a morphism of short exact sequences.

We denote $\operatorname{coker}(f) = N/\operatorname{im}(f)$ and similarly for f' and f''.

a. Show that the restriction of u to $\ker(f')$ is a linear map $u_1 \colon \ker(f') \to \ker(f)$ and that the restriction of v to $\ker(f)$ is a linear map $v_1 \colon \ker(f) \to \ker(f'')$. Show that

$$0 \to \ker(f') \xrightarrow{u_1} \ker(f) \xrightarrow{v_1} \ker(f'')$$

is exact.

b. Show that a induces by passing to the quotient a linear map $a_1 \colon \operatorname{coker}(f') \to \operatorname{coker}(f)$ and that b induces a linear map $b_1 \colon \operatorname{coker}(f) \to \operatorname{coker}(f'')$. Show that

$$\operatorname{coker}(f') \xrightarrow{a_1} \operatorname{coker}(f) \xrightarrow{b_1} \operatorname{coker}(f'') \to 0$$

is exact.

c. Let $m'' \in \ker(f'')$, let $m \in M$ such that $v(m) = m''$. Show that $f(m) \in \operatorname{im}(a)$; let $n' \in N'$ be such that $a(n') = f(m)$, and let $\widetilde{n}' \in \operatorname{coker}(f')$ be the class of n' modulo $\operatorname{im}(f')$ in $\operatorname{coker}(f')$.

Show that $\widetilde{n}'$ does not depend on the choices of m and n'. Deduce that the map $m'' \mapsto \widetilde{n}'$ is a well-defined R-linear map δ from $\ker(f'')$ to $\operatorname{coker}(f')$.

d. Show that the sequence

$$0 \to \ker(f') \xrightarrow{u_1} \ker(f) \xrightarrow{v_1} \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \xrightarrow{a_1} \operatorname{coker}(f) \xrightarrow{b_1} \operatorname{coker}(f'') \to 0$$

is exact.

e. Show that if f' and f'' are isomorphisms, then f is an isomorphism.

f. Show that if f' is surjective and f is injective, then f'' is injective.