D-MATH
HS 2021
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Exercise sheet 5

Commutative Algebra

(1) Let $m \geq 1$ be an integer and let $A_1, \ldots, A_m$ be non-zero rings. Show that

$$\dim(A_1 \times \cdots \times A_m) = \max(\dim(A_1), \ldots, \dim(A_m)).$$

(2)
a. Let R be a ring. Show that the set of non-principal ideals in R satisfies the conditions of Zorn's Lemma.

b. Let R be a UFD of dimension 1. Show that all prime ideals of R are principal, and conclude that R is a PID.

(3) Nagata's example of Noetheria domain of infinite dimension.
Let k be a field, and let $A = k[X_n : n \geq 1]$ be the polynomial ring in a countably infinite set of indeterminates. Let $m_1 > m_2 > \ldots$ be an increasing sequence of positive integers such that

$$m_{i+1} - m_i > m_i - m_{i-1}$$

for all $i > 1$. Let $\mathfrak{p}_i = (X_{m_i+1}, X_{m_i+2}, \ldots, X_{m_{i+1}})$ and let S be the complement in A of the union of the prime ideals $\mathfrak{p}_i$. Each $\mathfrak{p}_i$ is a prime ideal and therefore S is multiplicatively closed.

a. Show that $\mathrm{ht}(S^{-1}\mathfrak{p}_i) = m_{i+1} - m_i$ and deduce that $\dim S^{-1}A = \infty$.

To show that $S^{-1}A$ is noetherian, use the following.

Lemma. *Let A be a ring such that*

1. *for each maximal ideal $\mathfrak{m} \subseteq A$, the localization $A_{\mathfrak{m}}$ is noetherian;*
2. *for each $x \in A$, $x \neq 0$, the set of maximal ideals of A which contain x is finite.*

Then A is noetherian.

b. Show that $S^{-1}\mathfrak{p}_i$ are maximal ideals of $S^{-1}A$. To see that they are the only maximal ideals, use the *prime avoidence lemma*:
Let $J, I_1, \ldots, I_n$ be ideals of a ring A such that

$$J \subseteq \bigcup_{i=1}^{n} I_i.$$

If at least $n-2$ among the I_i are prime ideals, then there exists $i \in \{1, \dots, n\}$ such that $J \subseteq I_i$.

c. Show that $S^{-1}A$ satisfies condition 2 of the above lemma, and conclude.

4 a. Prove that the ring $R = \mathbb{C}[X,Y]/(Y^2 - X^3)$ is an integral domain of dimension 1.

b. Prove that R is not integrally closed (in its fraction field K).

c. Show that $p = (X-1, Y-1)/(Y^2 - X^3)$ is a well-defined maximal ideal in R.

d. Prove that the localization R_p is integrally closed (in its fraction field, which is also K).

5 Let $R = \mathbb{Z}[\sqrt{5}] = \mathbb{Z}[X]/(X^2 - 5)$.

a. Show that R is a noetherian integral domain of dimension 1 with fraction field equal to $K = \mathbb{Q}(\sqrt{5})$.

b. Show that R is not integrally closed.

c. Show that the integral closure of R in K is $\mathbb{Z}[(1+\sqrt{5})/2]$.