D-MATH
HS 2021
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Exercise sheet 6

Commutative Algebra

1 Let $R = \mathbf{Z}$ and let p be a prime number. We denote by M the R-module $\mathbf{Z}[1/p]/\mathbf{Z}$, contained in $\mathbf{Q}/\mathbf{Z}$.

a. If $N \subset M$ is a submodule such that $N \neq M$, then show that there exists $m \geq 0$ such that $1/p^n \notin N$ if $n \geq m$. Deduce that N is finite.

b. Show that M is an artinian module.

c. Prove that the length of M is infinite.

2 Let M, N be A-modules and let $f : M \to N$ be a morphism. Assume that N has finite length.

a. Show that $\ker f$ and $\operatorname{im} f$ have finite lengths and that

$$\ell(\operatorname{im} f) + \ell(\ker f) = \ell(M).$$

b. Assume that $N = M$. Show that the following conditions are equivalent: (i) f is bijective; (ii) f is injective; (iii) f is surjective.

c. Assume that M is artinian. Show that there exists an integer $n \geq 1$ such that $\ker(f^n) + \operatorname{im}(f^n) = M$.

3 a. Let M be an A-module of finite length. Show that the canonical morphism

$$M \longrightarrow \prod_{\mathfrak{m} \subseteq A \text{ max.}} M_{\mathfrak{m}}$$

sending $x \in M$ to the family of fractions $x/1$, is an isomorphism of A-modules.

b. Assume that A is artinian. Show that the canonical morphism

$$A \longrightarrow \prod_{\mathfrak{m} \subseteq A \text{ max.}} A_{\mathfrak{m}}$$

is an isomorphism of rings. Hence any commutative artinian ring is a product of local rings.

(4) Let A be a local, noetherian commutative ring and let $\mathfrak{m}$ be its maximal ideal. Let I be an ideal of A. Show that A/I has finite length if and only if there exists an integer $n \geq 0$ such that $\mathfrak{m}^n \subseteq I$.

(5) Let R be an artinian local ring with maximal ideal m and residue field $k = R/m$.

a. Prove that if every ideal in R is principal (warning! this does not mean that R is a PID, since R might not be an integral domain), then m/m^2 is a vector space of dimension ≤ 1 over k.

b. Show that $m = m^2$ if and only if R is a field.

c. Suppose that m/m^2 is a k-vector space of dimension 1.

  (i) Prove that m is a principal ideal.

  (ii) Let I be an ideal of R which is non-zero and different from R. Show that there exists $r \geq 0$ such that $I \subset m^r$ but I is not contained in m^{r+1}. (Hint: use the Jacobson radical.)

  (iii) Conclude that I is principal.

d. Show that if p is a prime number and $n \geq 1$, then $\mathbf{Z}/p^n\mathbf{Z}$ is an artinian local ring where every ideal is principal. When is it an integral domain?

e. Let k be a field and let $R = k[x^2, x^3]/(x^4)$. Show that R is an artinian local ring; determine the maximal ideal and show that the residue field is naturally isomorphic to k. Prove that m/m^2 has dimension 2 over k.