D-MATH
HS 2021
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Exercise sheet 7

Commutative Algebra

(1) a. Show that the fraction field of $\mathbb{C}[x,y]/(y^2-x^3-x)$ is not a purely transcendental extension of $\mathbb{C}$. (Hint: show first that if it were, then there would exist polynomials f and g in $\mathbb{C}[T]$ such that $f^2 = g^3 + g$.)

b. Show that it is *not* true that any field extension $K \subset L$ can be decomposed in $K \subset E \subset L$ where the extension E/K is algebraic and L/E is purely transcendental.

(2) Let $K \subset L \subset E$ be fields. Show that

$$\mathrm{trdeg}_K(E) = \mathrm{trdeg}_L(E) + \mathrm{trdeg}_K(L).$$

(3) Let K be an algebraically closed field and let R_1 and R_2 be finitely-generated K-algebras which are integral domains. Let $R = R_1 \otimes_K R_2$.

a. Show that R is a finitely-generated K-algebra and that $R \neq \{0\}$.

b. Show that any $f \in R$ can be expressed in the form

$$f = \sum_i a_i \otimes b_i$$

where $a_i \in R_1$ and $b_i \in R_2$, and where the b_i are linearly independent over K.

c. Let

$$f_1 = \sum_i a_i \otimes b_i \in R, \qquad f_2 = \sum_j c_j \otimes d_j \in R$$

be two elements of R, as in the previous question. Assume that $f_1 f_2 = 0$ in R. Let I_1 be the ideal in R_1 generated by (a_i) and I_2 the ideal generated by (c_j).

For any maximal ideal m of R_1, prove that either I_1 or I_2 is contained in m. (Hint: show that R_1/m is isomorphic to K and consider the morphism $\pi_m \otimes \mathrm{Id}_{R_2} \colon R \to R_1/m \otimes R_2 \simeq R_2$.)

d. Deduce that $I_1 \cap I_2 = \{0\}$.

e. If $f_1 \neq 0$, deduce that $I_2 = 0$, hence that $f_2 = 0$. (So that R is an integral domain.)

(4) Let $n \geq 1$ and $m \geq 1$ be integers and let f_1 ,..., f_m be elements of $\mathbb{Z}[X_1, \ldots, X_n]$. Let $\bar{\mathbb{Z}} \subset \mathbb{C}$ be the integral closure of $\mathbb{Z}$ in $\mathbb{C}$, and let $\bar{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

a. Show that for any $x \in \bar{\mathbb{Q}}$, there exists an integer $n \geq 1$ such that $nx \in \bar{\mathbb{Z}}$.

b. Let $k \geq 1$ be an integer and $(y_1, \ldots, y_k)$ in $\bar{\mathbb{Z}}^k$. Show that for any prime number p, there exists a maximal ideal $m \subset \mathbb{Z}[y_1, \ldots, y_k]$ such that $\mathbb{Z}[y_1, \ldots, y_k]/m$ is a finite extension of $\mathbb{F}_p$.

c. Let $k \geq 1$ be an integer and $(y_1, \ldots, y_k)$ in $\bar{\mathbb{Q}}^k$. Show that there exists an integer $N \geq 1$ such that the ring $\mathbb{Z}[1/N, y_1, \ldots, y_k]$ is an integral extension of $\mathbb{Z}[1/N]$ (which is the localization of $\mathbb{Z}$ at the element N), and that for any prime number p not dividing N, there exists a maximal ideal $m \subset \mathbb{Z}[1/N, y_1, \ldots, y_k]$ such that $\mathbb{Z}[1/N, y_1, \ldots, y_k]/m$ is a finite extension of $\mathbb{F}_p$.

d. Show that the system of equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0$$

has a solution in $\mathbb{C}^n$ if and only if it has a solution in $\bar{\mathbb{Q}}^n$.

e. For a prime number p, let $\bar{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$. Show that the system of equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0$$

has a solution in $\mathbb{C}^n$ if and only if, for all prime numbers p large enough, the system has a solution in $\bar{\mathbb{F}}_p^n$.