D-MATH
HS 2021
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Solutions 1

Commutative Algebra

1 a. Clearly G sends the identity Id_X to $\mathrm{Id}_{G(X)}$. To check the compatibility with respect to composition of morphisms, let

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

in $\mathscr{D}$. Then for all $x \in X$ one has

$$G(f \circ g)(x) = (f \circ g)(x).$$

On the other hand,

$$\mathbb{C}^{(X)} \xrightarrow{G(g)} \mathbb{C}^{(Y)} \xrightarrow{G(f)} \mathbb{C}^{(Z)}$$

$$x \longmapsto g(x) \longmapsto f(g(x)),$$

so

$$(G(f) \circ G(g))(x) = f(g(x))$$

and we get the desired equality.

b. Define

$$\begin{aligned} \Phi : \mathrm{Hom}_{\mathscr{C}}(G(X), V) &\longrightarrow \mathrm{Hom}_{\mathscr{D}}(X, F(V)) \\ (f : \mathbb{C}^{(X)} \to V) &\longmapsto (\Phi(f) : X \to V), \end{aligned}$$

where $\Phi(f)(x) = f(x)$ for all $x \in X$. The inverse of Φ is given by

$$\begin{aligned} \Psi : \mathrm{Hom}_{\mathscr{D}}(X, F(V)) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(G(X), V) \\ (g : X \to V) &\longmapsto (\Psi(g) : \mathbb{C}^{(X)} \to V), \end{aligned}$$

where $\Psi(g)(x) = g(x)$ for all $x \in X$ and extended by linearity.

2 a. No, because there are maps between non-trivial underlying sets of groups which are not morphisms of groups.

b. No, as above, there are maps between the underlying set of topological spaces which are not continuous.

c. No, consider the rings $X = Y = \mathbb{C}[x]$. The unit groups are both isomorphic to $\mathbb{C}^\times$. We call ϕ the map

$$\mathrm{Hom}_{(\mathrm{Rings})}(\mathbb{C}[x], \mathbb{C}[x]) \longrightarrow \mathrm{Hom}_{(\mathrm{Grp})}(\mathbb{C}^\times, \mathbb{C}^\times).$$

The valuation at 0 morphism in $\mathrm{Hom}_{(\mathrm{Rings})}(\mathbb{C}[x], \mathbb{C}[x])$ fixes the group of unity, as well as of course the identity map. Hence ϕ is not injective.

d. No, the reason lies in the cardinality. Take an infinite dimensional vector space. Then $\dim V^*$ is strictly larger. Hence there is no surjective map

$$\mathrm{Hom}_{\mathbb{C}}(V, W) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(W^*, V^*).$$

e. This is true, the map F is simply the transposition map, a bijection with inverse the transposition itself.

(3) a. No, take the empy set as a counterexample.

b. Yes, any set can be equipped with the discrete topology, for instance.

c. No, not every group is the group of unity of a ring. As a classical counterexample, let $\mathbb{Z}/5\mathbb{Z}$, the cyclic group of order 5.
Firstly, note that if R is any ring, then the characteristic of R is 2 or R has $-1 \neq 1$. So if a group G doesn't have a central element (i.e. it commutes with every element of R) of order 2, it can only arise as the group of units of a ring of characteristic 2.
Assume there exists R ring with R^* cyclic of order 5. Then R contains the field $\mathbb{F}_2$ and no any larger field, since such a field would contain a $(2^n - 1)$-th root of unity, but $(2^n - 1)|5$ iff $n = 1$.
Let $r \in R^* \simeq \mathbb{Z}/5\mathbb{Z}$. Then $r^5 = 1$ and the subring B generated by r is isomorphic to $\mathbb{F}[X]/(f)$ for some $f|X^5 - 1$. Now,

$$X^5 - 1 = (X - 1)(X^4 + X^3 + X^2 + 1)$$

as a product of irreducible polynomials. Thus either

- $f = X - 1$ and $B = \mathbb{F}_2$,
- $f = X^4 + X^3 + X^2 + 1$ and $B = \mathbb{F}_{2^4}$ with 15 units,
- $f = X^5 - 1$ and $B = \mathbb{F}_2 \times \mathbb{F}_{2^4}$ with 15 units,

contradicting the existence of just 5 units.

d. No, considerfor instance a vector space of countably infinite dimension over $\mathbb{C}$, like $\mathbb{C}[X]$. It cannot be the dual of any $\mathbb{C}$-vector space.

e. Yes, in the finite dimensional case we have a (non canonical) isomorphism $V \sim V^*$ for any vector space V.

4 a. and b. are straightforward checks.

c. Fully faithfullness:

$$\begin{aligned} M_{n,m}(\mathbb{C}) &\xrightarrow{F} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \\ A &\longmapsto (x \mapsto Ax). \end{aligned}$$

The bijectivity of the above map follows by linear algebra, the one to one correspondence between linear maps and matrices associated to fixed basis (e.g. canonical) of $\mathbb{C}^n$ and $\mathbb{C}^m$.
The functor F is also essentially surjective, since any vector space over $\mathbb{C}$ of dimension n is isomorphic to $\mathbb{C}^n$.

d. Define

$$\begin{aligned} G : (fdVec) &\longrightarrow \mathscr{C} \\ V =< \underline{v} >_{\mathbb{C}} &\longmapsto \dim V \\ (f : V =< \underline{v} >_{\mathbb{C}} \to W =< \underline{w} >_{\mathbb{C}}) &\longmapsto M_{\underline{w}}^{\underline{v}}(f) \in M_{\dim W, \dim V}(\mathbb{C}), \end{aligned}$$

where $\underline{v} = v_1, \ldots, v_{\dim V}$ is a fixed basis of V over $\mathbb{C}$ (same for W) and $M_{\underline{w}}^{\underline{v}}(f)$ is the matrix associated to f with respect to the basis $\underline{v}$ and $\underline{w}$.
The following diagrams commute for every $f : V \to W$ linear map and for every $n, m \in \mathbb{N}$, which means that

$$F \circ G \xrightarrow{\Phi} \mathrm{Id}_{(fdVec)}$$

and

$$G \circ F \xrightarrow{\Psi} \mathrm{Id}_{\mathscr{C}}$$

are natural transformations;

$$\begin{array}{ccc} \mathbb{C}^{\dim V} & \xrightarrow{M_{\underline{w}}^{\underline{v}}(f)} & \mathbb{C}^{\dim W} \\ {\scriptstyle \Phi(V)}\downarrow & & \downarrow{\scriptstyle \Phi(W)} \\ V & \xrightarrow{\quad f \quad} & W \end{array}$$

$$\begin{array}{ccc} n & \xrightarrow{M_{n,m}(\mathbb{C})} & m \\ {\scriptstyle \Psi(n)}\downarrow & & \downarrow{\scriptstyle \Psi(m)} \\ n & \xrightarrow{M_{n,m}(\mathbb{C})} & m \end{array}$$