D-MATH
HS 2021
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Solutions 2

Commutative Algebra

(1) Consider the morphism

$$\begin{aligned} \varphi : k &\longrightarrow \mathbb{Z}/p\mathbb{Z} \\ \overline{\frac{a}{b}} &\longmapsto [a][b]^{-1}. \end{aligned}$$

It is well-defined and actually an isomorphism, its inverse is

$$\begin{aligned} \psi : \mathbb{Z}/p\mathbb{Z} &\longrightarrow k \\ [a] &\longmapsto \overline{\frac{a}{1}}. \end{aligned}$$

Indeed,

$$\psi \circ \varphi\left(\overline{\frac{a}{b}}\right) = \psi([a][b]^{-1}) = \psi([ab^{-1}]) = \overline{\frac{ab^{-1}}{1}} = \overline{\frac{a}{b}}$$

and

$$\varphi \circ \psi([a]) = \varphi\left(\overline{\frac{a}{1}}\right) = [a].$$

(2)
a. Injectivity: Let $\frac{a}{1} = 0$ in $S^{-1}R$. Then there exists $s \in S$ so that $sa = 0$. But s is a unit, so $a = 0$.
Surjectivity: Let $\frac{a}{b} \in S^{-1}R$. Then $\varphi_S(ab^{-1}) = \frac{a}{b}$.

b. For every $b \in S$, since φ_S is surjective, there exists $r \in S$ so that $\frac{r}{1} = \frac{1}{b}$. This means that there is an $s \in S$ such that $s(rb-1) = 0$. So b is invertible with inverse $b^{-1} = sr$.

(3) The map ψ is given by the composition $\psi : R \overset{\iota}{\hookrightarrow} A \overset{\pi}{\to} B$.

a. For every $s \in S$, in B one has $[s][X_s] = 1$. Furthermore, f is in particular a ring morphism, hence

$$f \circ \psi(s) = f([s]) = f([X_s]^{-1}) = (f([X_s]))^{-1}.$$

So

$$(f \circ \psi(s))^{-1} = f([X_s]).$$

b. We want f so that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{g} & T \\ {\scriptstyle\psi}\downarrow & \nearrow {\scriptstyle f} & \\ B & & \end{array}$$

with $g(s) \in T^\times$ for all $s \in S$.
By using the universal property of R-algebras, we can extend g to a R-algebra morphism

$$\begin{aligned} g' : A &\longrightarrow T \\ X_s &\longmapsto g(s)^{-1}, \end{aligned}$$

$$\begin{array}{ccc} R & \xrightarrow{\iota} & A \\ \Big| & & \downarrow{\scriptstyle g'} \\ R & \xrightarrow{g} & T \end{array}$$

Furthermore, $I \subseteq \ker g'$, so there exists a map f so that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{g'} & T \\ {\scriptstyle\pi}\downarrow & \nearrow {\scriptstyle f} & \\ B & & \end{array}$$

By construction, one gets $f \circ \psi(r) = f([r]) = g(r)$ for all $r \in R$.

c. Part b. gives the surjectivity of the map $f \mapsto f \circ \psi$. For the injectivity, let $f \circ \psi = f' \circ \psi$. Then for all $r \in R$, $f([r]) = f'([r])$, but this means that $f = f'$.

d. The localization $S^{-1}R$ is an R-algebra via the canonical map $\varphi : R \to S^{-1}R$, and $\psi(S) \subseteq (S^{-1}R)^\times$. By the above, there exists $f : B \to S^{-1}R$ such that $\varphi = f \circ \psi$. Hence

$$f \circ \psi(r) = f([r]) = \frac{r}{1}$$

for all $r \in R$. Moreover,

$$f(X_s) = \varphi(s)^{-1} = \frac{1}{s}$$

for all $s \in S$.
Claim: f is an isomorphism.
Injectivity: if $\frac{r}{1} = 0$ in $S^{-1}R$, then there is an $s \in S$ with $sr = 0$. One can write $r = r(1 - sX_s)$, so $r \in I$.
Surjectivity: Let $\frac{r}{s} \in S^{-1}R$. Then

$$f([rX_s]) = f([r])f([X_s]) = \frac{r}{1} \cdot \frac{1}{s} = \frac{r}{s}.$$

(4) a. The ideal $\mathfrak{m}_A$ is the ideal generated by X. It is the kernel of the following morphism

$$\varphi : A \longrightarrow K$$
$$\sum_{n \geq 0} a_n X^n \longmapsto a_0.$$

Since $A/\mathfrak{m}_A \simeq K$, the ideal $\mathfrak{m}_A$ is maximal. It is the only maximal ideal of A, since $\mathfrak{m}_A = A \setminus A^\times$.

b. For every $p(X) = \sum_{k \geq 0} a_k X^k \in I$, let

$$n_p := \min\{k \geq 0 : a_k \neq 0\},$$

that is

$$p(X) = X^{n_p} q_p(X),$$

for some $q_p(X) \notin \mathfrak{m}_A$. Pick $p \in I$ so that $n = n_p$ is minimum. Then there is a $q \notin \mathfrak{m}_A$ such that

$$p(X) = X^n q(X).$$

Claim: $I = (X^n)$.
($\supseteq$) Since $q \in A^\times$, $X^n \in I$.
($\subseteq$) Let $r(X) = \sum_{k \geq 0} b_k X^k \in I$. By the choice of n, clearly $n_r \geq n$ and $b_k = 0$ for all $k < n$. That means that every non-zero term of r has an exponent of n or more, so we can factor out X^n and write r as

$$r(X) = X^n s(X),$$

for some $s \in A$. In particular, $r \in (X^n)$.

c. By the Cauchy product,

$$(1 + X)(1 - X + X^2 - X^3 + \dots) = 1,$$

so

$$(1 + X)^{-1} = \sum_{n \geq 0} (-1)^n X^n$$

in A.

(5) a. The polynomial f is a unit in $A[X]$ iff there is a $g = \sum_{i=0}^{m} b_i X^i \in A[X]$ such that $fg = 1$. Then $fg = \sum_{i=0}^{m+n} c_i X^i = 1$ with $c_i = \sum_{k+h=i} a_k b_h$. For $i = 0$, we have $a_0 b_0 = 1$, which implies that $a_0 \in A^\times$

For $i = m + n$ we obtain

$$a_n b_m = 0.$$

By multiplying c_{n+m-1} with a_n we have

$$a_n(a_{n-1}b_m + a_n b_{m-1}) = 0 \Longrightarrow a_n^2 b_{m-1} = 0.$$

Then

$$a_n^2 c_{n+m-2} = a_n^2(a_{n-2}b_m + a_{n-1}b_{m-1} + a_n b_{m-2}) = 0 \Longrightarrow a_n^3 b_{m-2} = 0$$

and so on (by induction). In particular

$$a_n^{m+1} b_0 = 0.$$

But b_0 is a unit, hence $a_n^{m+1} = 0$ and a_n is nilpotent.
Now, consider $f - a_n X^n$ and note that

$$(1 - a_n g X^n)(1 + a_n g X^n + (a_n g X^n)^2 + \cdots + (a_n g X^n)^m) \\ = 1 - (a_n g X^n)^{m+1} = 1,$$

so $1 - a_n g X^n$ is a unit. But f is also a unit, hence so is $f - a_n X^n$. By induction and by repeating the above argument we find that a_{n-1} is nilpotent and so on.

b. If $a_0, \ldots, a_n$ are nilpotent, so is f, since $f \in (a_0, \ldots, a_n)A[X]$ and the set of nilpotent elements is an ideal.
Conversely, let $k > 0$ such that $f^k = 0$, then clearly $a_0^k = 0$. Define

$$\begin{cases} f_0 = f \\ f_k = f_{k-1} - a_{k-1}X^{k-1} \quad \text{for } 1 \leq k \leq n-1. \end{cases}$$

Assume by induction that a_h is nilpotent for all $h \leq k-1$. Then $f_{k+1} = f - a_0 - a_1 X - \cdots - a_k X^k$ is nilpotent, so there is an ℓ such that $f_{k+1}^\ell = 0$, i.e.

$$X^{k\ell}(a_k + \cdots + a_n X^{n-k})^\ell = 0,$$

which implies $a_k^\ell = 0$.

c. Let $g\sum_{i=0}^{m} b_iX^i \in A[X]$, $g \neq 0$ be such that $fg = 0$ in $A[X]$. We can assume $b_0 \neq 0$ by observing that $Xgf = 0$ iff $gf = 0$. Pick also g of minimum degree.
In particular $a_nb_m = 0$, and clearly $(a_ng)f = 0$. Since $\deg(a_ng) < m$, by the choice of g, $a_ng = 0$. From

$$fg = a_0 + a_1Xg + \cdots + a_{n-1}X^{n-1}g \\ = a_0 + \cdots + a_{n-1}b_mX^{n-1+m} = 0$$

one has $a_{n-1}b_m = 0$. Again, $\deg(a_{n-1}g) < m$, so $a_{n-1}g = 0$.
By proceeding, obne obtains $a_{n-k}g = 0$ for all $k = 0, \ldots, n$. In particular $b_0a_k = 0$ for all $k = 0, \ldots, n$. So $b_0f = 0$.

In general,

$$\sqrt{(0)} = \bigcap_{\mathfrak{p}\text{ prime}} \mathfrak{p} \subseteq \bigcap_{\mathfrak{M}\text{ maximal}} \mathfrak{M} = \mathrm{J}(A[X]).$$

If $f \notin \mathfrak{M}$ for some $\mathfrak{M}$, then

$$\mathfrak{M} \subset (f) + \mathfrak{M}.$$

By the maximality of $\mathfrak{M}$, $(f) + \mathfrak{M} = (1)$. In particular there exist $g \in A[X]$, $h \in \mathfrak{M}$ such that

$$fg + h = 1,$$

so $1 - fg \in \mathfrak{M}$ is not a unit.

Then, if $f \in \mathrm{J}(A[X])$, for all $g \in A[X]$, $1-fg \in A[X]^{\times}$. Take $g = -X$. Thus

$$1 + fX = 1 + a_0X + \cdots \in A[X]^{\times}.$$

By part a, the coefficients $a_0, \ldots, a_n$ are nilpotent, and by b f is nilpotent.

(6) Define

$$\phi : S^{-1}(A[X]) \longrightarrow (S^{-1}A)[X]$$

by

$$\phi\Big(\frac{\sum a_iX^i}{s}\Big) = \sum_{i=0}^{\deg f} \frac{a_i}{s}X^i$$

for $\sum a_iX^i \in A[X]$, $s \in S$. Then ϕ is well-defined, since if $\sum a_iX^i/s = \sum b_iX^i/s'$, then there exists $s'' \in S$ such that

$$s''\Big(\sum_{i=0}^{n}(a_is' - c_is)X^i\Big) = 0$$

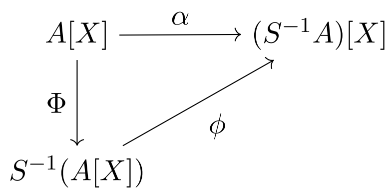
with $c_i = b_i$ if $i \leq m$ and 0 otherwise (assuming $n = \deg(\sum a_i X^i) \geq \deg(\sum b_i X^i) = m$). It turns out that

$$s''(a_i s' - c_i s) = 0$$

for $i = 0, \ldots, n$. Hence $a_i/s = b_i/s'$ in $S^{-1}A$ and

$$\phi\Big(\sum a_i X^i/s\Big) = \phi\Big(\sum b_i X^i/s'\Big).$$

It remains to show that ϕ is an isomorphism of rings, which is an easy check.
Alternatively, one can use the universal property of localization. For the ring morphism $\alpha : A[X] \to (S^{-1}A)[X]$, $\alpha(\sum a_i X^i) = \sum \frac{a_i}{s} X^i$ there is a unique ϕ such that the following diagram commutes



where Φ is the localization map, $\alpha = \phi \circ \Phi$.
On the other hand, since $S^{-1}(A[X])$ is a $S^{-1}A$-algebra, by the universal property of the polynomial ring, there is a unique morphism

$$\psi : (S^{-1}A)[X] \longrightarrow S^{-1}(A[X])$$

sending $1/X$ to $X/1$. It's now easy to check (it's enough to do for the indeterminate X) that ψ is the inverse of ϕ.