D-MATH
HS 2021
Prof. E. Kowalski

Solutions 3

Commutative Algebra

(1) a. The ideal I is finitely generated, pick generators $x_1, \dots x_m$ and $k_1, \dots, k_m \in \mathbb{N}$ so that $x_i^{k_i} = 0$ for all i. Let

$$\ell := \max_{i=1,\dots,m} k_i,$$

so that $x_i^{\ell} = 0$ for all i.
Let $x = \sum_{i=1}^m r_i x_i \in I$, $r_i \in R$. Then

$$x^{m\ell} \in (x_1^{i_1}, \dots, x_m^{i_m} : i_j \geq 0,\ i_1 + \cdots + i_m = \ell m) \subseteq (x_1^{\ell}, \dots, x_m^{\ell}) = 0.$$

Take $k = m\ell$.

b. Consider $R = \mathbb{C}[X_n : n \geq 1]/(X_n^n : n \geq 1)$. The nilradical of R is thge maximal ideal $I = (X_n : n \geq 1)$, but $I^n \neq 0$ for all $n \geq 1$.

(2) The proper ideals of $S^{-1}R$ are of the form $I(S^{-1}R)$, where I is an ideal of R, $I \cap S = \emptyset$. Note that if $I = (x_1, \dots, x_m)$ then $I(S^{-1}R) = \left(\frac{x_1}{1}, \dots, \frac{x_m}{1}\right)$, so it's finitely generated.
As a counterexample, pick any non-noetherian integral domain and it's fraction field (which is noetherian, since it's a field).
e.g. $R = \mathbb{C}[X_n : n \geq 1]$.

(3) Let $\varphi : R \longrightarrow M^n$ defined by $\varphi(x) = (xm_1, \dots, xm_n)$ for all $x \in R$, where $m_1, \dots, m_n$ are the generators of M over A. The kernel ofφ is precisely $(0 : M) = I$, so there is an embedding

$$R/I \hookrightarrow M^n.$$

Now, M is noetherian, then M^n is noetherian and so is R/I.

(4) a. We know $\mathbb{Q}$ is a $\mathbb{Z}$-module. By localizeing at the prime $p\mathbb{Z}$, one has that $\mathbb{Q}_{p\mathbb{Z}}$ has a structure of $\mathbb{Z}_{p\mathbb{Z}}$-module. But $\mathbb{Q}_{p\mathbb{Z}}$ is isomorphic to $\mathbb{Q}$.

b. The ring $\mathbb{Z}_{p\mathbb{Z}}$ is a local ring with maximal ideal $p\mathbb{Z}(\mathbb{Z}_{p\mathbb{Z}})$. Thus $J = p\mathbb{Z}(\mathbb{Z}_{p\mathbb{Z}})$. For all $\frac{r}{s} \in \mathbb{Q}$, write

$$\frac{r}{s} = \frac{p^R}{p^S} \cdot \frac{r'}{s'},$$

where $p \nmid r'$ and $p \nmid s'$. Then one can write

$$\frac{r}{s} = \frac{a}{b} \cdot \frac{r_1}{s_1}$$

with

$$\begin{cases} a = p^R,\ b = 1,\ r_1 = r',\ s_1 = s & \text{if } R \geq 1 \\ a = p,\ b = s',\ r_1 = r,\ s_1 = p^{S+1} & \text{if } R = 0. \end{cases}$$

In particular $\frac{a}{b} \in p\mathbb{Z}(\mathbb{Z}_{p\mathbb{Z}})$.

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a. This is a straightforward check.

b. Let $g(X) = a_t X^t + \cdots + a_0 \in \mathbb{Q}[X]$. Then

$$f(n) = (t + a_{t-1})n^{t-1} + \ldots$$

is a polynomial in n for all $n \in \mathbb{Z}$. So $f'(X) := (t+a_{t-1})X^{t-1} + \ldots$ is in R and $f'(n) = f(n)$ for all $n \in \mathbb{Z}$.

c. The binomial coefficients are integers. $\binom{n}{k}$ has integral values also for negative integers n, since

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

Let $f \in R$; we proceed by induction over $d := \deg f$.
If $d = 0$, then $f(X) = n \in \mathbb{Z}$, so

$$f(X) = n\binom{X}{0}.$$

Assume now that all polynomials in R of degree $< d$ can be written as a combination with coefficients in $\mathbb{Z}$ of $(f_k)_{k \geq 0}$. Write

$$f(X) = \frac{a_d}{b_d} X^d + \sum_{i=0}^{d-1} \frac{a_i}{b_i} X^i.$$

One has that

$$b_d f(X) - a_d d! \binom{X}{d}$$

is in R and its degree is $\leq d$. By induction, there are integers n_j so that

$$b_d f(X) - a_d d! \binom{X}{d} = \sum_{j=0}^{d-1} n_j \binom{X}{j}$$

$$\Longrightarrow f(X) = \frac{a_d}{b_d} d! \binom{X}{d} + \frac{1}{b_d} \sum_{j=0}^{d-1} n_j \binom{X}{j}.$$

Note that $f(0) = \frac{n_0}{b_d}$, so $b_d | n_0$; $f(1) = \frac{n_1}{b_d} + \frac{n_0}{b_d}$, so $b_d|n_1$. Also, $f(d-1) = \sum_{j=0}^{d-1} \frac{n_j}{b_d}\binom{d-1}{j}$, by induction one gets that $b_d|n_j$ for all $j = 0, \dots, d-1$. Since $f(d) = \frac{a_d}{b_d}d! + \frac{1}{b_d}\sum_{j=0}^{d-1} n_j\binom{d}{j}$ and the second summand is an integer, one gets that $\frac{a_d}{b_d}d!$ is an integer as well. So $f \in < f_k : k \geq 0 >_{\mathbb{Z}}$ and we conclude by induction.

d. $\mathbb{Z}$ is noetherian, but R is not finitely generated over $\mathbb{Z}$, so it is not noetherian.

(6) Consider the ascendent chain

$$\ker(u) \subseteq \ker(u^2) \subseteq \dots \subseteq \ker(u^n) \subseteq \dots$$

Since M is noetherian, there exists $n \geq 1$ such that $\ker(u^n) = \ker(n^{n+1})$. In particular, if $m \in M$ is so that $u(u^n(m)) = 0$, then $u^n(m) = 0$. Since u (and hence u^n) is surjective, we conclude that u is also injective.

(7) a. Let $\mathscr{F}_0$ be a totally ordered subset of $\mathscr{F}$. The maximum of $\mathscr{F}_0$ is $\bigcup_{I \in \mathscr{F}_0} I$. by Zorn's Lemmma, $\mathscr{F}$ has a maximal element P.

b. The ideal $P + (a) \supset P$, so $P + (a) \notin \mathscr{F}$ by the maximality of P. This means that $P + (a)$ is finitely generated,

$$P + (a) = (u_1, \dots, u_m, a)$$

where $u_1, \dots, u_m \in P$.
Also, $P \subseteq (P : a)$ and the inclusion is strict ($b \in (P : a)$ and $b \notin P$). As above, we have that $(P : a)$ is finitely generated,

$$(P : a) = (v_1, \dots, v_n)$$

with $v_1, \dots, v_n \in A$.

c. The inclusion "$\supseteq$" is clear by the above. On the other hand, if $x \in P$, we can write $x + a \in P + (a)$ as

$$x + a = \sum_{i=1}^{m} a_i u_i + ya,$$

where $a_i, y \in A$. Then $1 - y \in (P : a)$, since

$$(1 - y)a = \sum a_i u_i - x \in P;$$

hence

$$1 - y = \sum_{i=1}^{n} c_i v_i$$

for some $c_i \in A$. By multiplying both sides by a we get

$$x = \sum a_i u_i - \sum c_i v_i a.$$

Thus P is finitely generated, a contradiction.