D-MATH
HS 2021
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Solutions 4

Commutative Algebra

(1) a. Let $x = \sum_{i=1}^{2} x^i e_i$ and $y = \sum_{i=1}^{2} y^i e_i$. Then $x \otimes y = \sum_{i,j} x^i y^i e_i \otimes e_j$. By applying the isomorphism ϕ one has

$$\begin{aligned}\phi(x \otimes y) &= \sum_{i,j} x^i y^i \phi(e_i \otimes e_j) \\ &= x^1 y^1 f_1 + x^1 y^2 f_2 + x^2 y^1 f_3 + x^2 y^2 f_4.\end{aligned}$$

Hence $\phi(x \otimes y) = af_1 + bf_2 + cf_3 + df_4$ iff

$$\begin{aligned} x^1 y^1 &= a & \qquad x^1 y^2 &= b \\ x^2 y^1 &= c & \qquad x^2 y^2 &= d, \end{aligned}$$

which implies $ad = bc$. On the other hand, if $ad = bc$, $a, b \neq 0$, pick $x^1 = 1$, $x^2 = c/a = d/b$, $y^1 = a$ and $y^2 = b$. Similarly for the other possibilities for a, b, c, d.

b. Let $u = u_1 \otimes u_2 : \mathbb{R}^2 \otimes \mathbb{R}^2 \to \mathbb{R}^2 \otimes \mathbb{R}^2$. Then the matrix of u is given by the components of $u(f_i)$ with respect to the basis $(f_i)_i$:

$$\begin{aligned} u(e_1 \otimes e_1) &= u_1(e_1) \otimes u_2(e_1) = (1,0) \otimes (-1,2) = -f_1 + 2f_2 \\ u(e_1 \otimes e_2) &= 4f_1 + 3f_2 \\ u(e_2 \otimes e_1) &= -2f_1 + 4f_2 - 3f_3 + 6f_4 \\ u(e_2 \otimes e_2) &= 8f_1 + 6f_2 + 12f_3 + 9f_4. \end{aligned}$$

Therefore

$$u = \begin{pmatrix} -1 & 4 & -2 & 8 \\ 2 & 3 & 4 & 6 \\ 0 & 0 & -3 & 12 \\ 0 & 0 & 6 & 9 \end{pmatrix}.$$

(2) In $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ one has

$$2 \otimes 1 = 1 \otimes 2 \cdot 1 = 1 \otimes 0 = 0.$$

In general, for an A-module M we have a canonical isomorphism of A-modules

$$\begin{aligned} \phi : M \otimes_A A/I &\longrightarrow M/IM \\ m \otimes a &\longmapsto am \mod IM. \end{aligned}$$

So in our case

$$2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq 2\mathbb{Z}/(2)2\mathbb{Z} \simeq 2\mathbb{Z}/4\mathbb{Z};$$

via ϕ, $2\otimes 1$ is sent to 2 in $2\mathbb{Z}/4\mathbb{Z}$, which is not 0.

(3) a. The map $f \circ d : B \to M'$ is an A-derivation, since, using the B-linearity of f,

$$f(d(bb')) = f(bdb' + b'db) = bf(db') + b'f(db),$$
$$f(d(s(a))) = f(0) = 0$$

for all $b, b' \in B$, $a \in A$.

b. Consider the B-module

$$\Omega := B^{dB} = \bigoplus_{db \in dB} B.$$

Define

$$\Omega_{B/A} := \Omega/\Omega',$$

where Ω' is the B-submodule generated by the elements

$$d(bb') - bdb' - b'db,$$
$$d(b + b') - db - db',$$
$$d(s(a))$$

for all $b, b' \in B$, $a \in A$. Let

$$d_u : B \longrightarrow \Omega_{B/A}$$
$$b \longmapsto [db]$$

and let

$$f : \Omega_{B/A} \longrightarrow M$$
$$[db] \longmapsto db.$$

Then f is well-defined and has the desired properties. For the uniqueness, note that the universal derivation d_u is surjective, so f is determined by $f \circ d_u = d$.

To conclude, $\Omega_{B/A}$ is unique up to B-isomorphism: consider $M = \Omega_{B/A}$, $f = d_u$ and let $\Omega'_{B/A}$, $d'_u : B \to \Omega'_{B/A}$ be another solution of the universal problem. Let $f' : \Omega'_{B/A} \to \Omega_{B/A}$ be the B-linear map such that $f' \circ d'_u = d_u$. By the above property of $\Omega_{B/A}$, there is also a B-linear f so that $f \circ d_u = d'_u$. Therefore

$$f' \circ f \circ d_u = d_u,$$

which implies $f' \circ f = 1$ by the surjectivity of d_u.

(4) a. The map F is induced by the corresponding A- bilinear map, and it's given by

$$F(f_1 \otimes f_2) = (m_1 \otimes m_2 \mapsto f_1(m_1) \otimes f_2(m_2))$$

for every f_1, f_2, m_1, m_2 in the corresponding modules.

b. Since the vector spaces $\mathrm{Hom}_K(M_1, N_1) \otimes_K \mathrm{Hom}_K(M_2, N_2)$ and $\mathrm{Hom}_K(M_1 \otimes_K M_2, N_1 \otimes_K N_2)$ have the same dimension, it's enough to check the injectivity of F. Let $f_1 \in \mathrm{Hom}_K(M_1, N_1)$, $f_2 \in \mathrm{Hom}_K(M_2, N_2)$ such that

$$f_1(m_1) \otimes f_2(m_2) = 0 \quad \forall m_1 \in M_1,\ m_2 \in M_2.$$

Observe that if $f_i(m_i) \neq 0$ $(i = 1, 2)$, then $f_i(m_i)$ is part of a basis of N_i, so $f_1(m_1) \otimes f_2(m_2)$ is part of a basis of $N_1 \otimes_K N_2$, and it cannot be zero. On the other hand if $f_1(m_1) = 0$ or $f_2(m_2) = 0$, then the tensor product is 0. Since this holds for all m_1, m_2, we conclude that $f_1 = 0$ or $f_2 = 0$, so $f_1 \otimes f_2 = 0$.

c. In general, for an A-module M and an ideal I of A, we have

$$\mathrm{Hom}_A(A/I, M) \simeq (0 :_M I),$$

and by the A-freeness of A,

$$\mathrm{Hom}_A(A, M) \simeq M.$$

Hence we have an ismorphism

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}), \mathbb{Z}/4\mathbb{Z}) \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}))\\ \simeq 2\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})\end{aligned}$$

given by

$$\phi \otimes \psi \longmapsto \phi(1) \otimes \psi(1).$$

By a similar argument,

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}))\\ &\otimes_{\mathbb{Z}/4\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}))\\ &\simeq \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})\end{aligned}$$

given by

$$\alpha \longmapsto \alpha(1 \otimes 1).$$

Moreover,

$$2\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \simeq 2\mathbb{Z}/4\mathbb{Z}$$

via

$$2 \otimes 1 \longmapsto 2.$$

The corresponding map F' induced by F sends 2 to $2 \otimes 1$, which is 0 in $\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})$. Clearly $1 \otimes 1 \notin \mathrm{im}\, F'$.

d. As an $\mathbb{R}$-vector space, $\mathbb{C} \simeq \mathbb{R} \oplus \mathbb{R}$; since the tensor product commutes with finite direct sums we have isomorphisms of $\mathbb{R}$-vector spaces

$$\begin{aligned} \mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}^n &\simeq \mathbb{R}^{\oplus 2n} \otimes_{\mathbb{R}} \mathbb{C}^m \\ &\simeq (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}^m)^{\oplus 2n} \\ &\simeq (\mathbb{C}^m)^{2n} \\ &\simeq \mathbb{R}^{4mn}. \end{aligned}$$

5 a. Take $(f', f, f'') = (\mathrm{id}, \mathrm{id}, \mathrm{id}) : (M) \to (M)$ and the composition of $(f', f, f'') : (M) \to (N)$ and $(g', g, g'') : (N) \to (L)$ given by

$$(g', g, g'') \circ (f', f, f'') = ((g' \circ f'), (g \circ f), (g'' \circ f'')) : (M) \longrightarrow (L).$$

b. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle v \mod \ker v} & & \\ 0 & \longrightarrow & \ker v & \xrightarrow{i} & M & \xrightarrow{\pi} & \operatorname{coker} u & \longrightarrow & 0 \end{array}$$

Since u is injective, $\ker v = \operatorname{im} u$ and v is surjective, all the above vertical maps are isomorphisms.

6 a. Let $m' \in \ker f'$, then

$$f \circ u(m') = a \circ f'(m') = 0,$$

so $u_1(m') \in \ker f$. Same for v_1.
Furthermore, if $m' \in \ker f'$, then $v(u(m')) = 0$, since (M) is exact.

b. Since $a \circ f' = f \circ u$, for all $m' \in M'$,

$$a \circ f'(m') \in \operatorname{im} f.$$

Hence a induces a map

$$\operatorname{coker} f' \longrightarrow \operatorname{coker} f.$$

The exactness of (N) implies the claim.

c. Since

$$b \circ f(m) = f'' \circ v(m) = f''(m'') = 0,$$

one gets

$$f(m) \in \ker b = \operatorname{im} a.$$

Let $m'' = v(m) = v(m_1)$. Then $f(m_1) \in \operatorname{im} a$, so write $f(m_1) = a(n_1')$. The claim is that $\widetilde{n'} = \widetilde{n_1'}$ in $\operatorname{coker} f'$, equivalent to $n' - n_1 \in \operatorname{im} f'$:
$m - m_1 \in \ker v = \operatorname{im} u$, so $m - m_1 = u(m')$ for some $m' \in M'$.
$f(m - m_1) = f(m) - f(m_1) = a(n' - n_1') = f \circ u(m') = a \circ f'(m')$.
But a is injective, so $n' - n_1' = f'(m')$.

d. It remains to check the exactness at δ, i.e. that $\operatorname{im} v_1 = \ker \delta$ and that $\operatorname{im} \delta = \ker a_1$. We show that first equality, the second one is analogous.
($\subseteq$) Let $m'' \in \ker f''$, then $m'' = v(m)$ for some $m \in \ker f$. Since $f(m) = 0$, $n' \in \ker a$, so $n' = 0$ and $\delta(m'') = 0$.
($\supseteq$) Let $m'' \in \ker f''$ so that $\delta(m'') = 0$. Then for all $m \in v_1^{-1}(m'')$ there exists $m' \in M'$ such that

$$f(u(m')) = f(m).$$

Thus $m - u(m') \in \ker f$ and

$$\begin{aligned} v_1(m - u(m')) &= v_1(m) - v_1(u(m')) \\ &= v_1(m) \\ &= m''. \end{aligned}$$

e. If $\ker f' = \ker f'' = \operatorname{coker} f' = \operatorname{coker} f'' = 0$, from the exactness of the sequence in d, one has

$$\ker f = \operatorname{coker} f = 0.$$

f. If $\operatorname{coker} f' = \ker f = 0$, again by d one has the exact sequence

$$0 \longrightarrow \ker f'' \longrightarrow 0,$$

so $\ker f'' = 0$.