D-MATH
HS 2021
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Solutions 6

Commutative Algebra

(1) a. Let $N \subset M$, and write $N = N'/\mathbb{Z}$ for some $\mathbb{Z} \subseteq N' \subset \mathbb{Z}[1/p]$. Let

$$m := \max\{n \in \mathbb{N} : p^{-n}\mathbb{Z} \subseteq N'\}.$$

Then m is finite, if not $N' = \mathbb{Z}[1/p]$. By definition, N is generated over $\mathbb{Z}$ by p^{-m}.

Now, if $a \in \mathbb{Z}$, $a \geq p^m$, we can find $q \in \mathbb{Z}$ and $r \in \mathbb{Z}$ with $|r| < p^m$ such that

$$a = p^m q + r.$$

In particular,

$$ap^{-m} \equiv rp^{-m} \mod \mathbb{Z}.$$

This means, that a representative of the coset of ap^{-m} in N is given by an element rp^{-m}, with $|r| < p^m$. Hence N is finite.

b. Note that

$$p^{-n}\mathbb{Z} \subseteq p^{-m}\mathbb{Z}$$

if $m > n$. Every descendent chain is then of the form

$$p^{-n_0}\mathbb{Z} \supset p^{-n_1}\mathbb{Z} \supset \dots$$

where $n_0 > n_1 > \dots$. The latter chain of positive integers must stabilize, hence so the above chain.

c. The following is an infinite ascendent chain in M:

$$\mathbb{Z} \subseteq p^{-1}\mathbb{Z} \subseteq p^{-2}\mathbb{Z} \subseteq \dots$$

In particular, M is not noetherian, so its length is infinite.

(2) a. Since $\ker f$ is a submodule of M, $\ell(\ker f) \leq \ell(M) < \infty$. Moreover, by $\operatorname{im} f \simeq M/\ker f$ one has $\ell(\operatorname{im} f) = \ell(M) - \ell(\ker f) < \infty$.

b. Assume that f is injective. Since $\ell(M) < \infty$, in particular M is artinian. Then the following descendent chain stabilizes:

$$\operatorname{im} f \supseteq \operatorname{im} f^2 \supseteq \dots,$$

that is, there exists $t \geq 1$ such that $\operatorname{im} f^t = \operatorname{im} f^{t+1}$. Let $n \in M$ and let $m \in M$ such that $f^t(n) = f^{t+1}(m)$. Then

$$\begin{aligned} & f^t(n) - f^t(f(m)) = 0 \\ \Longrightarrow & f^t(n - f(m)) = 0 \\ \overset{f \text{ injective}}{\Longrightarrow} & n = f(m) \\ \Longrightarrow & n \in \operatorname{im} f. \end{aligned}$$

If f is surjective, consider the analogous ascendent chain

$$\ker f \subseteq \ker f^2 \subseteq \dots$$

and use the fact that M is noetherian.

c. Let n as above so that $\operatorname{im} f^n = \operatorname{im} f^m$ for all $m \geq n$. Let $m \in M$ and $m' \in M$ such that $f^{n+n}(m') = f^n(m)$. Then

$$m = (m - f^n(m')) + f^n(m')$$

and

$$f^n(m - f^n(m')) = f^n(m) - f^{n+n}(m') = 0,$$

so $m - f^n(m') \ker f^n$. Hence

$$M = \ker f^n + \operatorname{im} f^n.$$

3 a. Let $\Phi : M \to \prod_{\mathfrak{m} \subseteq A} M_{\mathfrak{m}}$. Note that if M is a simple module (i.e. $\ell(M) = 1$), $M \simeq A/\mathfrak{m}$, say, for some maximal ideal $\mathfrak{m}$ of A, then $M_{\mathfrak{m}} \simeq A/\mathfrak{m}$. Moreover, if $\mathfrak{m}' \neq \mathfrak{m}$, then

$$M_{\mathfrak{m}'} \simeq (A/\mathfrak{m})_{\mathfrak{m}'} \simeq A_{\mathfrak{m}'}/\mathfrak{m}_{\mathfrak{m}'} = 0,$$

since $\mathfrak{m} \not\subseteq \mathfrak{m}'$. In particular, if $\mathfrak{m}'$ and $\mathfrak{m}''$ are distinct maximal ideals, then $(M_{\mathfrak{m}'})_{\mathfrak{m}''} = 0$.
Now, let $n := \ell(M)$ and pick a decomposition series

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0.$$

By localizing at $\mathfrak{m}$, the fact that the quotients M_i/M_{i+1} are simple and the above remarks we get

$$(M_i/M_{i+1})_{\mathfrak{m}} = \begin{cases} M_i/M_{i+1} & \text{if } \mathfrak{m} =:_A (M_i/M_{i+1}) \\ 0 & \text{otherwise.} \end{cases}$$

From this, we see that $M_{\mathfrak{m}}$ has a decomposition series corresponding to the subseries of the one for M, obtained by keeping only

those $(M_i)_\mathfrak{m}$ such that $M_i/M_{i+1} \simeq A/\mathfrak{m}$.
Consider now a maximal ideal $\mathfrak{m}'$ and the localization of Φ:

$$\Phi_{\mathfrak{m}'} : M_{\mathfrak{m}'} \longrightarrow \prod_{\mathfrak{m}\subseteq A} (M_\mathfrak{m})_{\mathfrak{m}'} = (M_{\mathfrak{m}'})_{\mathfrak{m}'} = M_{\mathfrak{m}'}.$$

Then $\Phi_{\mathfrak{m}'} = \mathrm{id}_{M_{\mathfrak{m}'}}$ for every maximal ideal $\mathfrak{m}'$. In particular $\Phi_{\mathfrak{m}'}$ is an isomorphism of $A_{\mathfrak{m}'}$-modules for every maximal ideal $\mathfrak{m}'$; but the localization is a flat module, so the above implies that Φ is an isomorphism of A-modules (slogan: "being an isomorphism is a local property").

b. Since A is artinian, it has finite length, and there are only finitely many maximal ideals, so by part a we get an isomorphism of A-modules

$$\Phi : A \simeq \prod_{\mathfrak{m}\subseteq A} A_\mathfrak{m} \simeq \bigoplus_{\mathfrak{m}\subseteq A} A_\mathfrak{m}.$$

Since each map $A \to A_\mathfrak{m}$ is a morphism of rings, the isomorphism Φ is actually an isomorphism of rings.

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- Assume A/I has finite length. W still denote by $\mathfrak{m}$ the maximal ideal of A/I.
  One has that A/I is also artinian, so it has Krull dimension 0. In particular, the only prime ideal in A/I is $\mathfrak{m}$. Then
  $$J = \sqrt{(0)} = \mathfrak{m}.$$
  As in exercise 1 of Serie 3, there exists $n \geq 0$ so that $J^n = (0)$. Hence $\mathfrak{m}^n = (0)$ in A/I.
- Viceversa, let $\mathfrak{m}^n = (0)$ for some $n \geq 0$. One has the decomposition series in A/I
  $$A/I \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots \supseteq \mathfrak{m}^n = 0.$$
  Let $M_i := \mathfrak{m}^i/\mathfrak{m}^{i+1}$ for all $i = 0, \dots, n-1$. In particular, M_i is noetherian for all i. Since $\mathfrak{m}M_i = 0$, M_i is a $(A/I)/\mathfrak{m}$-vector space, so it is also artinian. By a ("reverse") induction, we show that $\mathfrak{m}^i$ is artinian for all i:
  If $i = n$, $\mathfrak{m}^n = 0$, so it is artinian.
  Assume now $\mathfrak{m}^{i+1}$ artinian. From the exact sequence
  $$0 \longrightarrow \mathfrak{m}^{i+1} \longrightarrow \mathfrak{m}^i \longrightarrow M_i \longrightarrow 0$$
  $\mathfrak{m}^i$ is artinian as well.
  From the exact sequence
  $$0 \longrightarrow \mathfrak{m} \longrightarrow A/I \longrightarrow (A/I)/\mathfrak{m} \longrightarrow 0$$
  we also get that A/I is artinian. Hence A/I is both artinian and noetherian, which implies that it has finite length.

(5) a. It is clear, since $\mathfrak{m}$ is principal, and a set of generators of $\mathfrak{m}/\mathfrak{m}^2$ over k is given by the generators of $\mathfrak{m}$ over R modulo $\mathfrak{m}^2$.

b. If $\mathfrak{m} = \mathfrak{m}^2$, by Nakayama's lemma, $\mathfrak{m} = 0$, so R is a field.

c. As a consequence of Nakayama's lemma, one has that

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \text{minimal number of generators of } \mathfrak{m}.$$

Then $\mathfrak{m}$ is principal. Pick $x \in R$ a generator.

Since R is artinian,

$$\mathfrak{m} = \sqrt{(0)} = J.$$

As before, there exists N so that $\mathfrak{m}^N = (0)$. Let

$$r := \max\{n \in \mathbb{N} : \mathfrak{m}^n \supseteq I\}.$$

Since $\mathfrak{m}^N = (0)$ and $I \neq (0)$, r is finite. By definition of r, $\mathfrak{m}^{r+1} \not\supseteq I$.

Thus there exist $y \in I$, $a \in R$ such that

$$y = ax^r \text{ and } y \notin (x^{r+1}).$$

In particular $a \notin \mathfrak{m} = (x)$, so it is a unit in R. We then have $x^r \in I$, so $\mathfrak{m}^r = (x^r) \subseteq I$. By the above, $I = (x^r)$.

d. The ring $\mathbb{Z}/p^n\mathbb{Z}$ is a finite local ring with maximal ideal $p\mathbb{Z}/\mathbb{Z}$. It is an integral domain if and only if $n = 1$, so a field.

e. R is artinian, since it is noetherian with just one prime ideal $(x^2, x^3)/(x^4)$:
Let $\wp \subseteq R$ be prime. Then $\wp = \wp'/(x^4)$, where $\wp' = (f_1, \dots, f_n, x^4) \subseteq k[x^2, x^3]$ prime, for some $f_1, \dots, f_n \in k[x^2, x^3]$. In particular, $x^4 \in \wp'$, so $x \in \wp'$, then

$$\wp' = (f_1, \dots, f_n, x)$$

and $(x^2, x^3) \subseteq \wp'$. But (x^2, x^3) is maximal in $k[x^2, x^3]$, since the morphism

$$\begin{aligned} k[x^2, x^3] &\twoheadrightarrow k \\ f &\mapsto f(0) \end{aligned}$$

has kernel (x^2, x^3). Then $\wp' = (x^2, x^3)$. Finally, modulo x^4 one has $\wp^2 = 0$, so $\dim_k(\wp/\wp^2) = 2$.