Prof. Rahul Pandharipande
Tim-Henrik Buelles

Student Seminar: Complex Algebraic Surfaces

ETH Zürich, HS 2021

Organizer: Tim-Henrik Buelles, HG J 16.4, buelles@math.ethz.ch

Time and Location: Fri 12 - 14, HG F 26.5

Goal: The aim of the seminar is to understand the Enriques classification of complex algebraic surfaces. We will see how techniques of algebraic geometry are applied to classify complex algebraic surfaces. Along the way we discuss invariants from cohomology and intersection theory and encounter important examples of varieties, such as ruled, abelian and K3 surfaces.

Prerequisites: We assume familiarity with the basic concepts of Algebraic Geometry, roughly in the amount of chapters II and III of Hartshorne's book [4].

Literature: This seminar will be heavily based on Beauville's book [2]. We recommend each participate to get a copy (available online), it is a joy to read.

Organization: One week before the talk, you should discuss with the organizer (e.g. following the Fri 12 - 14 session). The purpose is to discuss the objective and rough schedule of your talk. It is not meant to go into details or technicalities. Since the topic of this seminar is rather advanced, it should be a good idea to share your questions with the other participants and to discuss with each other.

Schedule of Talks

Remark: At the end of each chapter [2] you find *exercises* and *historical notes*. It is a good idea to incorporate some of them into your talk.

1 Divisors

Speaker: Tim Haupt
Date: 1 Oct 2021
Topics: Recall (without proof) the relation between Weil and Cartier divisors and line bundles [4, II 6], especially explain when the equalities $\mathrm{Pic}(X) = \mathrm{CaCl}(X)$ resp. $\mathrm{CaCl}(X) = \mathrm{Cl}(X)$ hold and define the canonical divisor of a smooth projective variety. Explain how divisors can be pulled back or pushed forward [2, I 1]. Define the degree of a line bundle on a curve [4, II 6.10]. Introduce linear systems and explain quickly the link to effective divisors and discuss [4, IV 1.2] and introduce rational morphisms [2, II 4-6], see also [4, II 7.1]. Recall the notions of ample and very ample line bundles and discuss their relations [4, pp. 150-156]. Prove Bertinis theorem [4, II 8.18] and deduce that any divisor on a smooth projective variety (over an algebraically closed field) is linearly equivalent to the difference of two smooth hypersurface sections.

2 Intersection theory and the Riemann-Roch theorem

Speaker: Gianni Gagliardo
Date: 8 Oct 2021
Topics: Prove Riemann–Roch theorem for curves [4, IV Thm. 1.3] and compute the degree of the canonical divisor of a curve [4, IV Ex. 1.3.3]. Prove [4, V Thm. 1.1] (see also [2, I.1-I.7] and the Riemann–Roch theorem for surfaces [4, V Thm. 1.6] (see also [2, I.12]). Recall the adjunction formula and discuss [4, V Prop. 1.5].

3 Blow-ups

Speaker: Fabian Roshardt
Date: 15 Oct 2021
Topics: Define the blow-up of a scheme in a closed subscheme [4, p. 162 ff.] and discuss the universal property of blow-ups [4, II Prop. 7.14]. Discuss the particular case of surfaces [2, II.1-8]. Describe the structure of a birational morphism between surfaces [2, II.11-12]. Briefly recall the Néron–Severi group as in [2, I.10] and explain [2, II.13].

4 Minimal models

Speaker: Yifan Zhao
Date: 22 Oct 2021
Topics: Introduce the notion of minimal surfaces [2, II.15-16]. Define the irregularity, the geometric genus and the plurigenera of a surface and prove that they are birational invariants [2, III.20]. Introduce the Kodaira dimension [2, VII]) and explain its role in the Enriques classification, see e.g. [3, Thm. 9.4]. Prove Castelnuovo's contractibility criterion [2, II.17], see also [6, 3.6].

5 Ruled and rational surfaces

Speaker: Quentin Roubaty
Date: 5 Nov 2021
Topics: The topic of this talk are surfaces which are birational to $C \times \mathbb{P}^1$ [2, III.1]. For non-rational C, chapter [2, III], for rational C chapter [2, IV](recall $\mathbb{P}^1 \times \mathbb{P}^1$ birational to $\mathbb{P}^2$). Define geometrically ruled surfaces and discuss [2, III.4] with a brief outline of the proof. Recall the fact [2, III.7] and finally arrive at [2, III.10]. Rational surfaces [2, IV] are a classical subject of study (explain why, have a look at the historical notes) and you can chose from a wealth of examples for illustration. Define Hirzebruch surfaces carefully and discuss their important role [2, III.15].

6 Castelnuovo's criterion for rationality

Speaker: Ana Pavlaković
Date: 12 Nov 2021
Topics: Present [2, V.1-10].

7 Albanese variety and minimal models for surfaces of non-negative Kodaira dimension

Speaker: Jonathan Hauenstein
Date: 19 Nov 2021
Topics: Introduce the Picard variety [3, p. 70-72] and the Albanese variety [3, p. 73-75], see also [2, V.11-16]. Prove that there is a unique minimal model for surfaces of non-negative Kodaira-dimension [2, V.15-19]. Finally state the theorem about the characterization of ruled surfaces [3, Thm. 13.2], see also [2, VI.2, VI.17].

8 Surfaces of Kodaira dimension 0

Speaker: Weisheng Wang
Date: 26 Nov 2021
Topics: Present [2, VIII.1-6]. The definition of bi-elliptic surfaces is [2, VI.19] and if time permits you can comment on examples, in particular [2, VI.16, VI.20].

9 K3 surfaces and Enriques surfaces

Speaker: Hussein Olama
Date: 3 Dec 2021
Topics: Show that K3 surfaces are simply connected [3, Thm. 10.3 (ii)] and give examples of K3 surfaces [2, VIII.8-11]. Study linear systems on K3 surfaces [2, VIII.13]. Show that a K3 surface is elliptic if and only if it contains a non-trivial divisor D with $D^2 = 0$, see [5, 2.3.13, 8.2.13]. Show that Enriques surfaces can be written as quotients of K3 surfaces (and vice versa) [2, VIII.17] and give examples of Enriques surfaces.

10 Surfaces of Kodaira dimension 1

Speaker: Lukas Bertsch
Date: 10 Dec 2021
Topics: Prove [2, IX.1, IX.2, IX.3] and give an example of an elliptic surface of Kodaira dimension 1. Introduce Kodaira fibration and give an example of a Kodaira fibration of general type [1, V.14].

11 Surfaces of general type

Speaker: Xiaowen Xie
Date: 17 Dec 2021
Topics: This talk should explore the class of surfaces of Kodaira dimension 2 [2, X], [1, V, VII]. You have some freedom to chose from various topics such as Kodaira fibrations, Godeaux surface, fake projective planes, geography of Chern numbers.

References

[1] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces*, second ed., vol. 4, Springer-Verlag, Berlin, 2004. MR 2030225

[2] Arnaud Beauville, *Complex algebraic surfaces*, second ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996. MR 1406314

[3] Lucian Bădescu, *Algebraic surfaces*, Universitext, Springer-Verlag, New York, 2001. MR 1805816

[4] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR 0463157

[5] Daniel Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016. MR 3586372

[6] Chris Peters, *An introduction to complex algebraic geometry - with emphasis on the theory of surfaces*, Notes.