Exercise Sheet 4

Exercise 1 (Related Vector Fields)**.** Let M, N be smooth manifolds and let $\varphi : M \to N$ be a smooth map. Recall that two vector fields $X \in \mathrm{Vect}(M)$, $X' \in \mathrm{Vect}(N)$ are called φ-related if

$$d_p\varphi(X_p) = X'_{\varphi(p)}$$

for every $p \in M$.

Show that $[X, Y]$ is φ-related to $[X', Y']$ if $X \in \mathrm{Vect}(M)$ is φ-related to $X' \in \mathrm{Vect}(N)$ and $Y \in \mathrm{Vect}(M)$ is φ-related to $Y' \in \mathrm{Vect}(N)$.

Exercise 2 (Leibniz Rule)**.** Let $A, B : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ be smooth curves and define $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ as the product $\varphi(t) := A(t)B(t)$. Show that

$$\varphi'(t) = A'(t)B(t) + A(t)B'(t)$$

for every $t \in (-\varepsilon, \varepsilon)$.

Exercise 3 (Some Lie Algebras)**.**

(a) Let M, N be smooth manifolds and let $f : M \to N$ be a smooth map of constant rank r. By the constant rank theorem we know that the level set $L = f^{-1}(q)$ is a regular submanifold of M of dimension $\dim M - r$ for every $q \in N$. Show that one may canonically identify

$$T_pL \cong \ker d_pf$$

for every $p \in L = f^{-1}(q)$.

(b) Use part a) to compute the Lie algebras of the Lie groups $O(n, \mathbb{R})$, $O(p, q)$, $U(n)$, $Sp(2n, \mathbb{C})$, $B(n)$ and $N(n)$ where $B(n)$ is the group of real invertible upper triangular matrices and $N(n)$ is the subgroup of $B(n)$ with only ones on the diagonal.

Exercise 4 (Easy Direction of Frobenius' Theorem)**.** Let M be a smooth manifold and let $\mathcal{D}$ be a distribution on M. Show that $\mathcal{D}$ is involutive if it is completely integrable.

Exercise 5 (Distributions and Lie Subalgebras)**.**

a) Let M be a smooth manifold, $X, Y \in \mathrm{Vect}(M)$ vector fields on M, and $f, g \in C^\infty(M)$ smooth functions. Show that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

b) Show that the Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Lie group G determines a left-invariant involutive distribution.

<u>Remark:</u> Part a) is not necessarily needed for part b).

Exercise 6 (Functions with values in immersed submanifolds)**.** Let M', M, N be smooth manifolds and let $\iota\colon N \hookrightarrow M$ be an injective immersion, i.e. ι is an injective smooth map whose differential is injective. Further, let $f\colon M' \to M$ be a smooth map with $f(M) \subseteq \iota(N)$.

Show that $\iota^{-1} \circ f\colon M' \to N$ is smooth if it is continuous.

Exercise 7 (Covering maps of Lie Groups)**.** Let G be a Lie group, let H be a simply connected topological space and let $p : H \to G$ be a covering map.

a) Show that there is a unique Lie group structure on H such that p is a smooth covering and a group homomorphism. Show also that the kernel of p is a discrete subgroup of H.

   Recall: $p : H \to G$ is a smooth covering if it is a topological covering which is smooth and such that each point in G has a neighbourhood U such that each component of $p^{-1}(U)$ is mapped diffeomorphically onto U by p.

b) Show that p is a local isomorphism of Lie groups and that dp is an isomorphism of Lie algebras when H is equipped with the Lie group structure from part a).

c) Let H, G be arbitrary Lie groups and let G be connected. Further, let $\varphi : H \to G$ be a Lie group homomorphism. Show that φ is a covering map if and only if $d\varphi : \mathfrak{h} \to \mathfrak{g}$ is an isomorphism.