Exercise Sheet 1

Exercise 1.(Unitary Operators):

Let $\mathcal{H}$ be a Hilbert space and $U(\mathcal{H})$ its group of unitary operators. Show that the weak operator topology coincides with the strong operator topology on $U(\mathcal{H})$.

Exercise 2.(Compact-Open Topology):

Let X, Y, Z be topological space, and denote by $C(Y,X) := \{f : Y \to X \text{ continuous}\}$ the set of continuous maps from Y to X. The set $C(Y,X)$ can be endowed with the *compact-open topology*, that is generated by the subbasic sets

$$S(K,U) := \{f \in C(Y,X) | f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.

Prove the following useful facts about the compact-open topology.

If Y is locally compact, then:

a) The evaluation map $e\colon C(Y,X) \times Y \to X, e(f,y) := f(y)$, is continuous.

b) A map $f : Y \times Z \to X$ is continuous if and only if the map

$$\hat{f} : Z \to C(Y,X), \hat{f}(z)(y) = f(y,z),$$

is continuous.

Exercise 3.(General Linear Group $\mathrm{GL}(n,\mathbb{R})$**):**

The general linear group

$$\mathrm{GL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} | \det A \neq 0\} \subseteq \mathbb{R}^{n\times n}$$

is naturally endowed with the subspace topology of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$. However, it can also be seen as a subset of the space of homeomorphisms of $\mathbb{R}^n$ via the injection

$$j\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{Homeo}(\mathbb{R}^n),$$
$$A \mapsto (x \mapsto Ax).$$

a) Show that $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ is a closed subset, where $\mathrm{Homeo}(\mathbb{R}^n) \subset C(\mathbb{R}^n,\mathbb{R}^n)$ is endowed with the compact-open topology.

b) If we identify $\mathrm{GL}(n,\mathbb{R})$ with its image $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ we can endow it with the induced subspace topology. Show that this topology coincides with

the usual topology coming from the inclusion $\mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$.
Hint: Exercise 2 can be useful here.

Exercise 4.(Isometry Group $\mathrm{Iso}(X)$**):**

Let (X,d) be a *compact* metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x,y) \quad \text{for all } x, y \in X\}.$$

Show that $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is compact with respect to the compact-open topology.

Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem.

Exercise 5.(p**-adic Integers** $\mathbb{Z}_p$**):**

Let $p \in \mathbb{N}$ be a prime number. Recall that the p-adic integers $\mathbb{Z}_p$ can be seen as the subspace

$$\left\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} : a_{n+1} \equiv a_n \,(\mathrm{mod}\ p^n)\right\}$$

of the infinite product $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}_p$ carrying the induced topology. Note that each $\mathbb{Z}/p^n\mathbb{Z}$ carries the discrete topology and $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is endowed with the resulting product topology.

a) Show that the image of $\mathbb{Z}$ via the embedding

$$\begin{aligned} \iota : \mathbb{Z} &\to \mathbb{Z}_p, \\ x &\mapsto (x \,(\mathrm{mod}\ p^n))_{n\in\mathbb{N}} \end{aligned}$$

is dense. In particular, $\mathbb{Z}_p$ is a compactification of $\mathbb{Z}$.

b) Show that the 2-adic integers $\mathbb{Z}_2$ are homeomorphic to the "middle thirds" cantor set

$$C = \left\{\sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N}\right\} \subset [0,1].$$

Exercise 6†.(Homeomorphism Group $\mathrm{Homeo}(X)$**):**

a) Let X be a *compact* Hausdorff space. Show that $(\mathrm{Homeo}(X), \circ)$ is a topological group when endowed with the compact-open topology.

b) Let $\mathbb{S}^1 \subset \mathbb{C}\backslash\{0\}$ denote the circle. Show that $\mathrm{Homeo}(\mathbb{S}^1)$ is not locally compact.
Remark: In fact, $\mathrm{Homeo}(M)$ is not locally compact for any manifold M.