Exercise Sheet 2

Exercise 1 (Transitive Group Actions)**.** Let G be a topological group, X a topological space and $\mu : G \times X \to X$ a continuous transitive group action, i.e. for any two $x, y \in X$ there is $g \in G$ such that $\mu(g, x) = g \cdot x = y$.

a) Show that if G is compact then X is compact.

b) Show that if G is connected then X is connected.

Exercise 2 (Examples of Haar Measures)**.** a) Let us consider the *three-dimensional Heisenberg group* $H = \mathbb{R} \ltimes_\eta \mathbb{R}^2$, where $\eta : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is defined by

$$\eta(x)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z + xy \end{pmatrix},$$

for all $x, y, z \in \mathbb{R}$. Thus the group operation is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and it is easy to see that it can be identified with the matrix group

$$H \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

Verify that the Lebesgue measure is the Haar measure of $\mathbb{R} \ltimes_\eta \mathbb{R}^2$ and that the group is unimodular.

b) Let

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}.$$

Show that $\frac{da}{a^2}\, db$ is the left Haar measure and $da\, db$ is the right Haar measure. In particular, P is *not* unimodular.

c) Let $G := \mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ denote the group of invertible matrices over $\mathbb{R}$. Let λ_{n^2} denote the Lebesgue measure on $\mathbb{R}^{n^2}$. Prove that

$$\mathrm{d}m(x) := |\mathrm{det} x|^{-n}\, \mathrm{d}\lambda_{n^2}(x)$$

defines a bi-invariant (i.e. left- and right-invariant) Haar measure on G.

d) Let $G = \mathrm{SL}_n(\mathbb{R})$ denote the group of matrices of determinant 1 in $\mathbb{R}^{n\times n}$. For a Borel subset $B \subseteq \mathrm{SL}_n(\mathbb{R})$ define

$$m(B) := \lambda_{n^2}\big(\{tg; g \in B, t \in [0,1]\}\big).$$

Show that m is a well-defined bi-invariant Haar measure on $\mathrm{SL}_n(\mathbb{R})$.

e) Let G denote the $ax+b$ group defined as

$$G = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} ; a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$$

Note that every element in G can be written in a unique fashion as a product of the form:

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$, which yields a coordinate system $\mathbb{R}^\times \times \mathbb{R} \leftrightarrow G$. Prove that

$$\mathrm{d}m(\alpha,\beta) = \frac{1}{|\alpha|}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

defines a left Haar measure on G. Calculate $\Delta_G(\alpha,\beta)$ for $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$.

Exercise 3 (Haar Measure and Transitive Actions)**.** Let G be a locally compact Hausdorff group and let X be a topological space. Suppose that G acts on X continuously and transitively. Let $o \in X$, and denote $\pi\colon G \to X, g \mapsto g \cdot o$. Further, let

$$H := \mathrm{Stab}(o) = \{h \in G \mid h \cdot o = o\}$$

be the stabilizer of o.

Suppose there is a continuous section $\sigma\colon X \to G$ of π, i.e. $\pi \circ \sigma = \mathrm{Id}_X$.

a) Show that $\psi\colon X \times H \to G, (x,h) \mapsto \sigma(x)h$ is a homeomorphism.

Hint: Find a continuous inverse!

b) Suppose there is a (left) Haar measure ν on H and suppose there is a left G-invariant Borel regular measure λ on X.

Show that the push-forward measure $\psi_*(\lambda \otimes \nu)$ is a (left) Haar measure on G.

c) Find a Haar measure on $\mathrm{Iso}(\mathbb{R}^2)$.

Exercise 4 ($\mathrm{Aut}(\mathbb{R}^n,+) \cong \mathrm{GL}(n,\mathbb{R})$)**.** For a topological group G, we denote by $\mathrm{Aut}(G)$ the group of bijective, continuous homomorphisms of G with continuous inverse. Consider the locally compact Hausdorff group $G = (\mathbb{R}^n, +)$ where $n \in \mathbb{N}_0$.

a) Show that $\mathrm{Aut}(G)$, i.e. the group of bijective homomorphisms which are homeomorphisms as well, is given by $\mathrm{GL}_n(\mathbb{R})$.

b) Show that $\mathrm{mod} : \mathrm{Aut}(G) \to \mathbb{R}_{>0}$ is given by $\alpha \mapsto |\mathrm{det}\alpha|$.

   Remark. By the definition given in the lecture $\mathrm{mod}(\alpha)$ is the unique positive real number such that $m(\alpha \cdot f) = \mathrm{mod}(\alpha)m(f)$ for all $f \in C_c(G)$, m left-Haar measure on G.
   This definitions may differ by an inverse from other definitions in the literature.

c) Prove that there exists a discontinuous, bijective homomorphism from the additive group $(\mathbb{R}, +)$ to itself.

Exercise 5 (Iterated Quotient Measures)**.** Let G be a locally compact Hausdorff group. Show that if $H_1 \leq H_2 \leq G$ are closed subgroups and H_1, H_2, G are all unimodular then there exist invariant measures dx, dy, dz on $G/H_1, G/H_2$ and H_2/H_1 respectively such that

$$\int_{G/H_1} f(x)dx = \int_{G/H_2} \left(\int_{H_2/H_1} f(yz)dz \right) dy$$

for all $f \in C_c(G/H_1)$.

Exercise 6 (No $\mathrm{SL}_2(\mathbb{R})$-invariant Measure on $\mathrm{SL}_2(\mathbb{R})/P$)**.** Let $G = \mathrm{SL}_2(\mathbb{R})$ and P be the subgroup of upper triangular matrices. Show directly that there is no (non-trivial) finite G-invariant measure on G/P.

<u>Hint:</u> Identify $G/P \cong \mathbb{S}^1 \cong \mathbb{R} \cup \{\infty\}$ with the unit circle and consider a rotation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and a translation

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$